the librarian *supplement*

loo vajpayee and nostalgia

randall shaw the international mathematical olympiad

mathews large numbers

ip phantom pain

brown second century judæo-christian relations

number three, volume two

Write for *The Librarian*.

Contact the editor for more information.

joshua.loo@westminster.org.uk

The Library
librarians@westminster.org.uk
Chair, Library Committee
jonny.heywood@westminster.org.uk
Assistant Chair, Library Committee
isky.mathews@westminster.org.uk
Editor, *The Librarian*
joshua.loo@westminster.org.uk
***The Librarian* Online**
`https://librarian.cf`
Scientific correspondence, *The Librarian*
isky.mathews@westminster.org.uk
Mathematic correspondence, *The Librarian*
benedict.randallshaw@westminster.org.uk
Puzzle-related correspondence, *The Librarian*
isky.mathews@westminster.org.uk
benedict.randallshaw@westminster.org.uk

Contents

News

Those who wish to join the Library Committee should send an email to the Chair, Jonny Heywood, whose email is listed above; they should include availability for desk duty, and some explanation as to why the applicant desires to join the committee.

The Librarian as usual finds itself in the position of supplicant; articles are welcome. We should be particularly grateful to receive book reviews.

As the start of the school year moderately significant occasions we consider it appropriate to issue a welcome and/or an expression of goodwill and/or wishes for good luck.

Errata

Cary Godsal spotted that the website managed to omit the last word of from the sentence '*The Librarian* is typeset in a Bembo-like font using 'LaTeX 2ε'; this is because Pandoc, the conversion tool, was unable to cope with the LaTeX 2ε `\LaTeXe`; this issue may have been fixed by the time this issue went to print.

Due to an accidental triplication of folders, the previous issue of *The Librarian* appeared thrice on the homepage of the website, thus creating several dead links. The error has been rectified.

The disclaimer previously concluded 'unless there is some explicit indication to the otherwise'; the error has been corrected, as is visible below.

About *The Librarian*

Note: articles in *The Librarian* do not necessarily reflect the views of any entity, notwithstanding any impression created to the contrary, unless there is some explicit indication otherwise.

For the purposes of clarity, 'any entity' refers, *inter alia*, without prejudice as to the paragraph above, to authors, those connected with them, *The Librarian*, the editors thereof, the Library Committee, the members, the Chair and the Assistant Chair thereof, the library, the librarians, the school, the teachers and other employees thereof, the members of the board and the board as a collective entity thereof, and any other body created, constituted or controlled thereby or in possession of any control thereof.

The Librarian is the publication of the Library Committee of Westminster School. The existence of a Library Committee dates back to at least December 1879, when the editor of *The Elizabethan* replied to a letter on the 'disgraceful' state of the books in the library, that '[s]ome years ago a regular library committee was in existence'. The present state of the library is far removed from its state in the late 1870s; the employment of four librarians, the Library Committee and the general interest of the rest of the school have all combined to ensure that there is little danger of a lapse into disrepair. The Library Committee broadly exists to support the work of the librarians; some examples of this support include the conveying of pupil views to the librarians, direct support (e.g., in desk duty, and charitable activities), and the publication of *The Librarian*.

Some find that they are unwilling to 'go all the way to the library'. Consequently, *The Librarian* offers a subscription service. Readers may email the editor, with a specified destination, which must either be an email or a physical location. This is, of course, free, as is *The Librarian* in general. Issues are occasionally uploaded to `https://librarian.cf`, which is likely to be increasingly frequently updated. The athletically blessed are encouraged to make the journey to the library. The physical location in most circumstances must be in the school; we do not rule out alternative arrangements, but most would be insufficiently feasible.

The Librarian is typeset in a Bembo-like font using LaTeX 2ε. Authors retain copyright of their works; rights to everything else remain *The Librarian*'s unless the context makes this repugnant.

We encourage submissions of all kinds. These include, but are not limited to, articles, reviews, letters, puzzles, short stories, poems, compositions, and answers to problems in the Adventures in Recreational Mathematics series. Submissions may be sent to the editor. Readers may also place notices in *The Librarian*, by prior arrangement with the editor.

As *The Librarian* is the publication of the Library Committee, it was initially principally concerned with books, and reviews of works that one might find in the library. However, many other articles have also been found. Thus *The Librarian Supplement* was formed, to house those articles that do not fit in the traditional scope of *The Librarian*. The division is to some extent arbitrary, and so we encourage readers to read both.

A sufficient number of mathematic and scientific articles are published in *The Librarian Supplement* to require the services of Benedict Randall Shaw and Isky Mathews respectively, so correspondence on those subjects (except letters) should be directed to them.

Letters

Policy on letters We urge readers not to write any letters that would make necessary the formulation of a policy more extensive than this.

Submissions Readers may submit letters to the editor, or to Room 5, Lower Corridor, College. Identification is unnecessary; impersonation is prohibited.[0]

Note on letters Letters are reproduced as they were sent. Consequently, *The Librarian* is not responsible for, *inter alia*, any intellectual inadequacies or errors of grammar, syntax, or orthography that may appear in letters. We are, of course, responsible for any errors of our own creation.

Guidance We suggest that readers in need of advice consult the October 1874 number of *The Elizabethan*.

Sir,

May I again fill your august columns with complaint? Whilst Deus misereatur appears thrice in the almanack diary, Ad te levavi appears eight times.

I remain, &c.,

Dies iræ

Sir,

Dangerous rumours on the wind suggest that *The* (esteemed) *Librarian* and the infamous *Pink* are the products of the same shadowy hand. It is thought that the pbulic bickering and stylistic polarity between these two publications are deliberate ploys to mislead the Westminster community and indeed Under Masters past and present. Please put minds at rest—unless 'Joshua Loo' is simply the ingenious creation of the anonymous anarchist collective: performative pandæmonium we fear more than ever as the new year arrives.

Petronius

Editor—growing sections of the school comunity are appalled at the pugnaciously gendered language throughout your own and other publications across the school. As a trendsetting force in the periodical world of Westminster, please lead the way down a wider, more inclusive road.

Arria

Note: writers of letters probably should not be too worried about their choice of address, since so few letters are sent that almost all will be published.

Sir,

Can you confirm that the Library Drawing Room is now available for hire for private 'raves'? Please advise on booking.

Trimalchio

0. The effective realise of this policy is that any attempt to identify an author (whether the true author or not) will precipitate a process of verification, which will normally consist of some attempt to contact the author; to expedite this, you may wish to provide some set of plausible contact details, such as a school email or verified pgp public key.

Friday 7th, 4:30

Lecture Room

In a world with a no-deal Brexit,
This House believes that London should exit the UK.

Atal Bihari Vajpayee

Joshua Loo

Consider the funeral of a Nehru-Gandhi or equivalent in stature. The police lathi charge[0] the crowds, almost reflexively—rarely is there a threat. The army do not need to dirty themselves by obstructing the crowds, and instead march in front and behind the cortège, where former associates gather; slogans are raised in honour of the departed. Then, the pyre; Hindu almost always—all three Nehru-Gandhis and Gandhi were, after all—the departed's ashes are then scattered in the appropriate river.

Vajpayee's death and funeral remind us, as those funerals might have, many of the pathologies of modern India. It appears that several articles prematurely reported his death.[1] The media's epistemic fragility is not inherently worse in India than in any other state, for the same factors prompt it—a desire to attract consumers, cut-throat competition to be the first, cost-cutting, and a public who do not appear to discriminate even when faced with the consequencies of their tendencies to choose those at the forefront of such unhelpful trends. It is thus that Arnab Goswami et al. manage to turn what might have taken a column inch or two into sensational stories lasting many minutes—'this happened, now, for more on this, our correspondent vaguely related to this…[the correspondent greets the anchor]—indeed, this has just happened, a stunning victory for…' (at this point the political party supported by the television channel, normally the BJP, is mentioned, followed by a similar tautologic interview). However, because announcers have taken to emulation of Goswami, who obtains respite from shouting by occasionally letting his interviewees speak for a moment, but have no interviewees, they must pause, thus elongating the process further. Thus column inches produce double or triple the number of logorrhœic minutes. Even Doordashan, once home to sedate state-approved monologues, has changed. Only All India Radio's news bulletins[2] have remained largely unchanged.

A number of other claims also circulated. Arvind Kejriwal, Aam Admi Party (AAP) Chief Minister of Delhi was accused of celebrating his birthday immediately after Vajpayee's death;[3] in reality, he stopped his birthday celebrations immediately after hearing the news. The Prime Minister was accused of smiling when visiting the hospital in a picture that was later shown to have been taken in a visit a few hours before.[4]

Yet there is more. In Aurangabad, Syed Mateen Sayyad Rashid, an All India Majlis-e-Ittehadul Muslimeen (AIMIM, or MIM) corporator was beaten for opposing a condolence motion. He has now been arrested under §§ 153, 153A and 294 of the Indian Penal Code (IPC).[5] Cases under §§ 323, 506 and 147 have been registered against the Bharatiya Janata Party BJP corporators who thrashed Rashid, but no arrests have been made so far. Samjay Kumar, an assistant professor at Mahatma Gandhi Central University, is said by regional news channels to have been dragged out of his house before being stripped and thrashed.[6] Swami Agnivesh, an activist best known for his opposition to bonded labour, was beaten outside the BJP's headquarters, where Vajpayee's body lay in state.[7] There have also been incidents at Jawaharlal Nehru University (JNU).[8]

It is tempting to view these incidents as part of a broader anti-democratic malaise. In January, four justices of the Supreme Court held a press conference, in which they attacked the allocation of cases by the Chief Justice, Dipak Misra.[9] Shashi Tharoor alleges that a BJP victory in coming Lok Sabha elections would cause India to become a 'Hindu Pakistan', incurring the wrath of at least one barrister, who has dragged him to the Calcutta High Court.[10] The Communist Party of India (Marxist)—CPI (M)—says that '[t]hese four years of Modi Government has seen[sic]…severe undermining of institutions of parliamentary democracy and independent constitutional authorities[sic]'[11], whilst the Communist Party of India (CPI) say in a draft political resolution that '[a]lmost all the democratic constitutional institutions including Parliament are being downgraded. To destroy the basic ethos of Constitution and our secular democratic polity, worst form of authoritarianism is be-

0. It is difficult to ascertain the number of injuries caused, because this is never reported. Either the police manage to charge towards large crowds waving around lathis without injuring anyone, or, more plausibly, it is felt that coverage could be insensitive to the departed, and invite police raids. Most other nations seem able to avoid lathi charges. Contrast the live fire of their colleagues in, for example, the Ugandan police, and the plasticised and mechanically administered violence of Western police forces. Unfortunately, the police are not limited to lathi charges; in recent months, the police killed nearly a dozen in Tamil Nadu after protests over a copper plant. Police firing, however, seems less common in modern India than it was under the Raj, or in other ex-colonies—most topically, Uganda.
1. Pooja CHAUDHURI. *How Media Outlets Announced Former PM Atal Bihari Vajpayee's Death Prematurely*. Aug. 16, 2018. URL: https://www.altnews.in/atal-bihari-vajpayee-death-national-media/.
2. *NEWS ON AIR*. URL: http://airnews.nic.in/ (visited on 08/20/2018).
3. Pooja CHAUDHURI. *Did Arvind Kejriwal Celebrate His Birthday after Former PM Atal Bihari Vajpayee's Death?* Aug. 17, 2018. URL: https://www.altnews.in/did-arvind-kejriwal-celebrate-his-birthday-after-former-pm-atal-bihari-vajpayees-death/.
4. Arjun SIDHARTH. *Viral Photo of PM Modi with Doctors- Is It from AIIMS?*. Aug. 17, 2018. URL: https://www.altnews.in/viral-photo-of-pm-modi-with-doctors-is-it-from-aiims/ (visited on 08/20/2018).
5. "AIMIM Corporator Arrested for Opposing Condolence Motion on Vajpayee, BJP Members Booked after Ruckus". In: *The Hindu. Other States* (Aug. 18, 2018). In collab. with PTI. ISSN: 0971-751X. URL: https://www.thehindu.com/news/national/other-states/aimim-corporator-arrested-for-opposing-condolence-motion-on-vajpayee/article24725076.ece (visited on 08/21/2018).
6. "Opposition Leaders, Intellectuals Slam Police Raids, Arrest of Activists". In: *Hindustan Times* (2018-08-30T00:38Z). URL: https://www.hindustantimes.com/india-news/opposition-leaders-intellectuals-slam-police-raids-arrest-of-activists/story-Yq1LIk1gTAjNQ8xoE2upxH.html (visited on 08/31/2018).
7. "Swami Agnivesh Assaulted on Way to Pay Homage to Atal Bihari Vajpayee in New Delhi". In: *The Indian Express* (Aug. 17, 2018). URL: https://indianexpress.com/article/india/swami-agnivesh-assault-ddu-marg-new-delhi-5311309/ (visited on 08/21/2018).
8. Students at the JNU are extremely politically involved—often leaning to the left. The quality of teaching is also often very high. Consequently, the Sangh Parivar and its allies (see footnote 14) in the Indian media have launched various campaigns against the university, even resulting in arrests. The university is 'littered with condoms', 'anti-national', full of infiltrators, seditious, and so on.
9. Dhananjay MAHAPATRA and Amit Anand CHOUDHARY. "Four Top Judges Revolt against CJI; Supreme Court on Trial". In: *The Times of India* (Jan. 13, 2018). URL: https://timesofindia.indiatimes.com/india/four-top-judges-revolt-against-cji-supreme-court-on-trial/articleshow/62480926.cms (visited on 07/15/2018).
10. "Shashi Tharoor Summoned by Kolkata Court over His 'Hindu-Pakistan' Remark". In: *The Indian Express* (July 14, 2018). URL: https://indianexpress.com/article/india/shashi-tharoor-summoned-by-kolkata-court-over-his-hindu-pakistan-remark-5259076/ (visited on 07/15/2018).
11. *Report on Certain Developments Since the 22nd Party Congress[Sic]*. July 14, 2018. URL: https://cpim.org/documents/report-certain-developments-22nd-party-congress (visited on 07/15/2018).

ing practiced.[sic]'[12] Rahul Gandhi, President of the Congress Party, says that the BJP are 'attacking the Constitution'.[13]

The media cannot escape scrutiny either. A *Cobrapost* investigation 'expose[d] many Indian media houses willing to peddle Hindutva[14]' for money.

It is certainly true that India has seen significant change since present ministry took office in late May 2014. Of the most recent state elections, the BJP's National Democratic Alliance (NDA) has won over 20, and the rest 11. In the nine years since the establishment of the Unique Identification Authority of India (UIDAI)[15], over 90% of the population have registered for an Aadhaar number[16]; the Planning Commission that wrote the notice on its establishment has now been abolished.

It is not clear what the ruling party intend to do with their hegemony. Civil service reform is in the air: recent proposals include lateral entry to the Indian Administrative Service for Joint Secretaries[17], and the allocation of cadres after foundational courses[18]. They certainly, however, desire to retain it (perhaps unsurprisingly).

12. *DRAFT POLITICAL RESOLUTION FOR 23RD CONGRESS.* National Council, Communist Party of India. URL: https://drive.google.com/file/d/1D3Tl9I4WWTxBj6thcHMREN8mIYdQCTNL/view?usp=sharing (visited on 07/15/2018).
13. "Constitution of the Country Is under Threat from the BJP: Rahul Gandhi". In: *The Times of India* (Dec. 28, 2017). URL: https://timesofindia.indiatimes.com/india/constitution-of-the-country-under-threat-rahul-gandhi/articleshow/62277333.cms (visited on 07/15/2018).
14. Hindutva is derived from an orthographically identical Romanisation of the Sanskrit and Hindi word for the state of being Hindu. At present, however, it refers to a Hindu nationalist ideology, whose core tenets appear to be Islamophobia and Hindu exceptionalism. It is difficult to determine other views. Capital's temporary alliance with the cow means that, for now, Hindu nationalists prop up an essentially neoliberal macroeconomic agenda. At the same time, however, their xenophobic streak, which was offended even by the possibility of a foreign-born Sonia Gandhi becoming Prime Minister, cannot but have been at the very least worried by liberalisation on foreign investment. Similarly, Hindu nationalists attack Muslims for their supposedly illiberal social views. Gay people, therefore, are part of a tolerant Hindu-Indian fabric so long as Islamic organisations attack them; once such criticisms are out of the spotlight, they are 'against Indian culture'. China, perhaps the greatest threat to India's national security, is now praised in Hindu nationalist circles, for detaining millions of Muslims. On the other hand, Indian Communists are Chinese plants. Hindutva may not be an entirely empty ideology, but what passes for Hindutva at present often appears to be. Hindutva is also described as the ideology of the BJP, which is, in turn, part of a broader 'Sangh Parivar'. The Sangh Parivar refers to a set of Hindu nationalist organisations; their organisational disunity arises both because it is expedient (parts of it were banned by Nehru, for example) and due to differences in policy. Vajpayee, a BJP minister, is often said to have insufficiently focused on Hindutva policies by ideologues in the RSS, a separate organisation that is not a political party.
15. Planning Commission GOVERNMENT OF INDIA. "Gazette of India, 2009, No. 324". In: *The Gazette of India* (Feb. 14, 2009), p. 138. URL: http://archive.org/details/in.gazette.2009.324 (visited on 07/16/2018).
16. Aman SHARMA. "Almost Every Adult Has an Aadhaar Now". In: *The Economic Times. Politics and Nation* (July 15, 2018). URL: https://economictimes.indiatimes.com/news/politics-and-nation/almost-every-adult-has-an-aadhaar-now/articleshow/65001223.cms (visited on 07/16/2018).
17. Moushami Das GUPTA. "Bureaucrats Wary of Lateral Entry in Govt Service as Deadline for Applications Looms". In: *The Hindustan Times* (June 30, 2018). URL: https://www.hindustantimes.com/india-news/unease-among-bureaucrats-over-lateral-entry-as-deadline-for-govt-s-ad-for-joint-secretary-looms/story-WApB0BSVGoPpOWqiZIwazI.html (visited on 07/16/2018).
18. Neeraj CHAUHAN. "Govt Wants Civil Servants to First Undergo Foundation Course before Service/Cadre Allocation - Times of India". In: *The Times of India* (May 20, 2018). URL: https://timesofindia.indiatimes.com/india/govt-wants-civil-servants-to-first-undergo-foundation-course-before-service/cadre-allocation/articleshow/64247938.cms (visited on 07/16/2018).

Party	States[19]
NDA—BJP	Arunachal Pradesh
	Assam
	Chhattisgarh
	Goa
	Gujarat
	Haryana
	Himachal Pradesh
	Jharkhand
	Madhya Pradesh
	Maharashtra
	Manipur
	Rajasthan
	Tripura
	Uttar Pradesh
	Uttarakhand
NDA—other	Andhra Pradesh
	Bihar
	Meghalaya
	Nagaland
	Sikkim
Congress, coalition partners	Karnataka
	Mizoram
	Puducherry
	Punjab
Independents	Delhi
	Kerala
	Odisha
	Tamil Nadu
	Telangana
	West Bengal
Governor's rule	Jammu and Kashmir[20]

Table 1: State governments by party.

Party	Population[21]
NDA—BJP	655
NDA—other	158
Congress, coalition partners	91
Independents	291

Table 2: Populations ruled by political alliances, in millions.

Party	GDP[22]
NDA—BJP	871
NDA—other	145
Congress, coalition partners	152
Independents	500

Table 3: GDP (US$, bn) of states ruled by political alliances.

18. *About Chief Minister of India, List of CM in India.* URL: http://www.elections.in/government/chief-minister.html#info_id7 (visited on 08/21/2018)
20. Jammu and Kashmir has been excluded from the other tables, since it is nominally under governor's rule, a coalition between a party advocating 'self-rule' and the BJP having fallen apart. But a recent appointee was a BJP official, and the governor generally follows the centre's orders, so the state is really under the effective BJP control.
21. Calculated from the 2011 census and the table above.
22. Calculated from Policy Commission GDP data 2014-5 and table 1.

Most recently, five intellectuals were arrested in coördinated raids across India. It was alleged that they were plotting to assassinate the Prime Minister, and that they had extensive links with banned Maoist groups.[23,24] In a pleasing show of judicial vitality, however, the Supreme Court has moved them from police detention to house arrest.[25]

It must, however, have been with little hope that those few dissidents remaining near the end of Emergency continued to avoid normalisation. After all, neither from from the public sphere nor any of the branches of government had sustained and effective opposition arose. It was Gandhi's decision to call elections, which she then lost, that caused her to lose power. Sonia Gandhi, her daughter in law, in an interview with New Delhi Television (NDTV), said that, at heart, she was a 'democrat'; Emergency must, therefore, have felt somewhat intolerable to her. Regardless, it seems probable that Emergency could quite easily have continued, if not for some combination of liberal sensibilities and overconfidence. The distribution of authority in Modi's India can provide little comfort to the victims of its ideology, for it also reduces the risk of such miscalculation.

Suppose that this hypothesis is true. Three interpretations in view of Vajpayee are open to us. Vajpayee may, first, have actively worked in the opposite direction. However, he may, second, have little relation to them, or, even, third, have instigated changes in the opposite direction.

The second possibility can be discounted immediately, from first principles. Vajpayee was the Prime Minister. Let us take press freedom as an example. Press freedom exists when the press are not unduly pressured. Inaction, therefore, is almost always positive; we worry about governmental action, because that suggests undue influence. Ergo, if Vajpayee did nothing, he must have been a rather good Prime Minister for press freedom. The same logic applies in the case of political dissidence, tribal, caste, and environmental activism, religious freedom, and so on. If he did nothing, the first interpretation of Vajpayee's legacy is true. If he did very much, either the first or third must be true. In no case can the second be true.

The first view appears to be that of the Indian establishment. Bhavna Vij Aurora wrote in *Outlook*[26] that '[i]n an era when a conciliatory tone is seen as a sign of weakness, Vajpayee rose above the rest to emerge as the biggest consensus builder'; supposedly, he had a 'benign inclusivity'. On Kashmir, 'even …Modi was forced to acknowledge in his Independence Day Address that the best solution lay in the three powerful words enunciated by Vajpayee'—"'Kashmiriyat, Insaniyat and Jamuriyat"'—which 'managed to disarm even the hardliners among the separatists'. But it is also mirrored in other parts too. The CPI (M), for example, wrote[27]:

> The Polit Bureau of the Communist Party of India (Marxist) expresses its grief at the death of former Prime Minister Shri Atal Behari Vajpayee.
>
> Shri Vajpayee had a distinguished political career in parliament, in government and as Prime Minister of India.
>
> As a political leader he commanded respect of all sections.

The CPI wrote[28]:

> The National Secretariat of the Communist Party of India expresses its deep-felt condolences at the passing away of former Prime Minister Atal Bihari Vajpayee at 5.05 p.m. at the All India institute of Medical Sciences on August 16, 2018. He was admitted to the AIIMS as his health deteriorated on June11 following a urinary tract infection.
>
> He has been suffering from a lower respiratory tract infection and kidney-related ailments.
>
> He was the first non-Congress prime minister to complete a full five-year term in office. He was earlier Prime Minister once for 13 days and another time for 13 months. He entered politics in the 1940s and was a highly respected leader loved by one and all. He was fondly called Atalji. He tried to rule on consensus. Atalji's death is a loss to the nation.
>
> The Party sends its condolences to his family members.

There are a number of common themes in these tributes; it is probably worth reading a few more[29], individual synpopses

23. Maoism in India can be said to take two forms: first, that of extraparliamentary struggle, and, second, an electoral strain. It is often linked with tribal and lower-caste struggle. Many but not all Maoist groups have been banned. Maoist guerillas are difficult to defend, on two grounds. First, they regularly behave coercively towards the very villagers they claim to defend. Second, they invite the response of a state that seems inherently incapable of behaving without violence. One need not go as far as Naipaul to conclude that those who truly are Marxist guerillas should probably be jailed in some cases. The exceptions would include areas where there is genuine local participation in Maoist guerilla movements. Another problem is that Maoism itself does not have a particularly good track record.
24. Kai Schultz and Suhasini Raj. "'Activists in Shackles': Indians Denounce Arrests as Crackdown on Dissent". In: *The New York Times. World* (Aug. 31, 2018). ISSN: 0362-4331. URL: https://www.nytimes.com/2018/08/30/world/asia/india-activists-arrests.html (visited on 08/31/2018).
25. The same confused tendencies of Hindutva can also be observed in the reaction to these arrests in Hindu nationalist groups. When judges complained to the Chief Justice about the allocation of politically sensitive cases, the Chief Justice became something of a hero; many urged the Prime Minister to extend his term. On the other hand, the judiciary are now part of a Naxalite alliance of intellectuals and metropolitan élites, conspiring to undermine India. It seems that the ultimate good is the promotion of the political interests of the Sangh Parivar. Quite what, therefore, would cause its supporters to dissent is unclear, given that their support appears not to be instrumental, but axiomatic.
26. Bhavna Vij-Aurora. "Atal Bihari Vajpayee: Poet By Instinct, Politician By Accident". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/website/story/atal-bihari-vajpayee-poet-by-instinct-politician-by-accident/315115 (visited on 08/22/2018).
27. *Atal Behari Vajpayee*. Aug. 16, 2018. URL: https://www.cpim.org/pressbriefs/atal-behari-vajpayee (visited on 08/28/2018).
28. *CPI Condoles Passing Away of Former Prime Minister A.B. Vajpayee*. Aug. 17, 2018. URL: https://www.facebook.com/cpofindia/posts/cpi-condoles-passing-away-of-former-prime-minister-ab-vajpayeethe-national-secre/2148762051825018/ (visited on 08/28/2018).
29. L.K. Advani. "'Atalji Suggested We Be Ready For The Police'". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/atalji-suggested-we-be-ready-for-the-police-excerpt-from-lk-advanis-book/300513 (visited on 08/28/2018); A.S. Dulat. "A Hawk's Friend and A Dove's Dream". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/vajpayee-politician-par-excellence-by-asdulat/300517 (visited on 08/28/2018); K.N. Govindacharya. "I Called Vajpayee 'Face of BJP', Media Made It 'Mukhota'". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/i-called-vajpayee-

of which would not be particularly useful. In this view, the most important aspects of Vajpayee's rule were, first, an increase in economic and mlitary strength—in particular, nuclear tests conducted and roads constructed at his behest are considered, second, his political credentials—his desire for compromise, his ability to unite different groups, and so on, third, good character—he is variously a 'poet', 'gentle giant', and so on, and, fourth, his political competence. Over the three Vajpayee ministries, a sufficient number of events must have occurred to provide ample evidence for all these assertions, at least taken in isolation.

But that is not all that there was to Vajpayee. It is easy to overlook the darker tendencies of the Indian state under his rule, but it is also unwise.

Outlook is possibly the best publication in India. Its articles are detailed and well-written—an increasingly rare tendency in a media environment that appears to have stolen all the worst aspects of the Anglo-American tabloid press and subtracted copy-editing. It appears to be far less afraid than other publications of publishing controversial stories and criticising government policy. It also seems to have a slightly stronger connexion to the truth than some other publications—it may be that the smaller size of its staff causes it to be slower, and therefore reduces the frequency with which it is caught out, but, whether deliberately or not, accuracy is the principal beneficiary. In the middle of the third Vajpayee ministry, *Outlook* published a story about corruption in the Prime Minister's Office (PMO). India has a famously corrupt reputation. It is not clear when this started, for administration in India used to be the preserve of largely British civil servants, and Sir Cyril Radcliffe managed to get through his set of Reith lectures without sounding obviously two-faced whilst maintaining that, after a series of reforms, the Imperial Civil Service seemed to be moderately honest, but the scandals surrounding V.K. Krishna Menon are one plausible candidate.[30] For reasons that are not entirely clear, the PMO took great exception to the article, and commenced a series of income tax raids on the owner of *Outlook*. Whatever one says about the Congress Party, they seem to have managed not to show the truth of allegations against them by launching raids on hostile media outlets. Vinod Mehta, then editor of *Outlook*, responded characteristically sardonically: '[g]overnment officials are compounding their folly by lying through their teeth. Sadly, even the lies are incompetent.'[31] Eventually, they gave in; the interrogations became too much, as Ajith Pillai recounts in *The Wire*[32]. It is difficult to believe that Vajpayee was not involved.

The story that caused so much distress to Vajpayee was titled 'Rigging the PMO'[33]. It is clear that corruption reached the very heart of the government, at least during the third Vajpayee ministry. According to the article, the Telecoms ministry was bypassed by a 'Group of Ministers' charged with overseeing liberalisation[34] in that sector. 'What makes the GoM [group of ministers] unique is that it clears decisions despite objections from the ministries concerned.' These questionable organisational practices seemed to be linked to a series of dubious decisions elsewhere: 'the provision of counter-guarantee for the Reliance group's Rs 20,000-crore Hirma power roject in Orissa'[35], 'direct dealings with power firms and international financiers' instead of 'competitive bidding', a 'decision piloted by the PMO to allow Fixed Service Providers to provide mobile services through wireless telephony', the two beneficiaries being 'Reliance and Himachal Futuristic Communications Ltd.', clearance of a '$1-billion[sic] Oman fertiliser project... which involves India purchasing 1.65 million tonnes of urea at a mandated price for the next 15 years', despite 'serious reservations' from the 'finance ministry and Public Investment Board', who argued that 'India was not in any pressing need[sic] to expand its urea utilisation', charges whose effective realise was that the government paid 'in instalments for a facility' that a private company had constructed 'for its own export and import activity'—a suspicious subsidy, and so on. To whose benefit, one might ask, were these decisions? It seems apparent that these were not intended to increase organisational efficiency and thus streamline fundamentally beneficial programes for the people of India; rather, they were to the benefit of industrialists—the 'Hinduja brothers', Reliance, et al. At the heart of these allegations were three bureaucrats: Brajash Mishra, the Principal Secretary to Vajpayee, N.K. Singh, an official in the PMO, and Ranjan Bhattacharya, Vajpayee's son-in-law.

face-of-bjp-media-made-it-mukhota/300515 (visited on 08/28/2018); V.P. Malik. "Vajpayee: Statesman In Peace And In War". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/vajpayee-statesman-in-peace-and-in-war-by-gen-retd-vp-malik/300516 (visited on 08/28/2018); D. Raja. "Vajpayee Was The Right Man In The Wrong Party". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/vajpayee-was-the-right-man-in-the-wrong-party-by-d-raja/300514 (visited on 08/28/2018); Shatrughan Sinha. "'Yeh Bihari Babu Hain Toh Main Bhi Atal Bihari Hoon'". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/yeh-bihari-babu-hain-toh-main-bhi-atal-bihari-hoon/300512 (visited on 08/28/2018); Yashwant Sinha. "All Doors In Kashmir Opened...Because Of Him". In: *Outlook* (Aug. 27, 2018). URL: https://www.outlookindia.com/magazine/story/all-doors-in-kashmir-openedbecause-of-him-yashwant-sinha-on-vajpayee/300511 (visited on 08/28/2018); Saubhadra Chatterji and Prashant Jha. "Atal Bihari Vajpayee Was so Much More than a Great Orator, Says Pranab Mukherjee". In: *Hindustan Times* (Aug. 20, 2018). URL: https://www.hindustantimes.com/india-news/atal-bihari-vajpayee-was-so-much-more-than-a-great-orator-says-pranab-mukherjee/story-aBYAoCSwfMal3AuzusKoEN.html (visited on 08/29/2018); MJ Akbar. "Atal Bihari Vajpayee: His Weapon Was the Word, Not the Sword". In: *The Times of India* (Aug. 17, 2018). URL: https://timesofindia.indiatimes.com/india/atal-bihari-vajpayee-his-weapon-was-the-word-not-the-sword/articleshow/65432723.cms (visited on 08/29/2018); Press Trust of India. "Vajpayee a Towering Figure Who Stood for Democratic Values: Sonia Gandhi - Times of India". In: *The Times of India* (Aug. 16, 2018). URL: https://timesofindia.indiatimes.com/india/vajpayee-a-towering-figure-who-stood-for-democratic-values-sonia-gandhi/articleshow/65430516.cms (visited on 08/29/2018).

30. It might be added that corruption and brutality are orthogonal. Amritsar, if anything, showed how competent the British apparatus was; Dyer could not be dissuaded from destroying the illusion of a civilising mission by such frailties as corruption or human sensibilities.

31. Vinod Mehta. "This Is An Income Tax Raid?" In: *Outlook* (May 31, 2001). URL: https://www.outlookindia.com/website/story/this-is-an-income-tax-raid/211800 (visited on 08/31/2018).

32. Ajith Pillai. *Outlook's Owner Was Raided During Vajpayee's Time, but the Media Response Was Quite Different*. July 11, 2018. URL: https://thewire.in/media/outlooks-owner-raided-vajpayees-time-media-response-quite-different (visited on 09/01/2018).

33. Ibid.

34. The term 'liberalisation' is commonly used in relation to the dismantling of the so-called Nehruvain 'licence raj'. It is probably true that the dismantling of state monopolies could be described as 'liberalisation', provided that their replacements were not crony capitalists of the sort that currently dominate the Indian economy, and, as the article indicates, quite possibly dominated it under Vajpayee as well.

35. In the ex-Raj, it is common to use 'crore' to denote ten million and 'lakh' to denote a hundred thousand. In this case, 20,000 crores is equal to something of the order of US$ 3,000,000, though this does not account for inflation or fluctuation in exchange rates. Whilst this may not sound particularly significant, when one accounts for inflation, and the growth of the Indian economy over the nearly two decades since the allegations, the sum would perhaps have been three or four times more significant than it sounds now. Even in the United States, such sums would not be regarded as entirely insignificant. Another factor that must be considered is that the Indian formal economy is very small, and so tax revenues are fairly minimal.

In a story two months later, *Outlook* doubled down on the accusations. It seems that the construction of roads also involved corrupt practices. 'In one stretch—Panagarh to Palsit in West Bengal—the lowest bid was, strangely, Rs 600 crore, even though the Infrastructure Development Finance Company had estimated the cost to be Rs 250 crore. What makes the NHAI deal lucrative is that the firms are paid in full under the annuity scheme and do not have to recover their investment through charging tolls.'

All were implicated. Ranjan and the PMO appeared to give an 'extra push' to a road construction programme. 'The Rs 20,000-crore fast-track Hirma power project being put up by Reliance is the other in which a finance ministry bureaucrat says Bhattacharya has shown particular interest. On the question of counter-guarantee, the PMO had been pushing the Reliance case. Says a former bureaucrat who till recently held a key economic post in the Vajpayee government, "Interested lobbies are very clever. They go through Ranjan. There is a base law in logic. If it rains, then some grass will sprout. If the grass has sprouted, there is the probability that it has rained."'

It may be objected that Vajpayee's reign saw significant economic growth. But so too did the Singh and Rao ministries, for which there is little nostalgia. It was the perceived manner in which this growth was achieved that seems to matter to the nostalgic.

In 2002, widespread communal violence that plausibly would be better classified as a pogrom or even ethnic cleansing took the lives of several hundred Hindus and several thousand Muslims.[36] A train 'carrying back from Faizabad (Uttar Pradesh) nationalist Hindu activists who had travelled to Ayodhya to build a temple dedicated to the god Ram on the ruins of the Babri Masjid'[37,38] was attacked in Godhra[39]. The passengers 'chanted Hindu nationalist songs and slogans throughout the entire voyage, all the while harassing Muslim passengers. One family was even made to get off the train for refusing to utter the *kar sevaks*' war cry: "Jai Shri Ram!" (Glory to Lord Ram!) ...a Muslim shopkeeper was also ordered to shout "Jai Shri Ram!" He refused, and was assaulted until the *kar sevaks* turned on a Muslim woman with her two daughters. One of them was forced to board the train before it started going again.' Then the violence escalated: 'one of the passengers pulled the emergency chain. ...[T]he train came to a halt in the middle of a Muslim neighborhood[sic] inhabited by Ghanchis, a community from which many of the Godhra street vendors hail. Anywhere from 500 to 2,000 of them, depending on the sources, surrounded the coach occupied by the *kar sevaks* and attacked it with stones and torches. Coach S-6 caught fire, killing 57 people.'[40] Despite the spontaneous nature of the murders, Modi declared that it was a "pre-planned violent act of terrorism"; he further 'called together police officials at his home and gave them orders not to put down the Hindus who would inevitably react to the Godhra attack'. Vajpayee attempted to remove Modi from his post, but failed.

It is not clear what Vajpayee's true views on communal relations in India are. But this is man who said of Muslims that wherever they 'are living, they don't want to live in harmony'—'[t]hey don't mix with the[sic] society', and 'are not interested in living in peace'. Of the Gujarat riots, he asked '[w]ho lit the fire?', ignoring the rôle of the state in the orchestrated escalation of violence that occurred.[41] Even more disturbing is his rôle in the demolition of the Babri Masjid. In December 1992, 150,000 Hindu nationalists assembled near the Babri Masjid, allegedly the site of the birth of Lord Ram. After someone climbed onto the mosque and waved a saffron flag (a Hindutva symbol), the crowd assembled proceeded to break through a police cordon and destroy the building. Thus began several months of communal violence that took the lives of perhaps two thousand Muslims and Hindus.[42] Vajpayee said in an interview on NDTV broadcast a little while after the demolition that 'what happened...was very unfortunate'—'it should not have happened'. A video taken before the demolition shows that Vajpayee delivered a speech to the assembled *kar sevaks* (Hindu nationalists).

> Sharp stones are emerging from the ground ...the ground will have to be levelled...if a *yagya* begins, there will have to be some construction.
>
> I don't know what will happen tomorrow. ...I have been instructed not to visit Ayodhya and I shall abide by this. I have no wish to visit the court and get tried in any case.

These are curious remarks. They are quite clearly in tension, possibly terminal, with Vajpayee's remarks on NDTV. Certainly, they are not the words of a man who would regret 'construction'. But even a rôle as ambiguous as this must be regarded as ignoble in an affair so sordid.

Crucial to the image that the BJP presently projects is the idea that it alone, amongst the hodgepodge of rising regional parties (the TMC, DMK, AIDMK, and so on) and falling national parties (the Communists and Congress), represents competence and order. This is why it arrests intellectuals on the charge that they are 'urban Naxals'[43]. It is also the motive for recent moves to replace currency, supposedly intended to reduce money laundering and achieve a number of other changing objectives. Also crucial is the propagation of the narrative that it is only the BJP who operate cleanly. They were helped in this respect by the incompetence of the Congress Party before the 2014 General Elections. It is inescapably the case, however, that Vajpayee, now in death a key part of BJP propaganda, was neither clean and competent nor capable of maintaining order—either that, or he played a deliberate part in the instigation of communal violence at and after the events of Ayodhya. What is more surprising and unfortunate is that so many others have repeated this revisionist view of his legacy.

36. Christophe Jaffrelot. *Communal Riots in Gujarat: The State at Risk?* Working Paper 17. Heidelberg: South Asia Institute, Department of Political Science, University of Heidelberg, July 2003, p. 21. URL: http://archiv.ub.uni-heidelberg.de/volltextserver/4127/1/hpsacp17.pdf.
37. Ibid., p. 3.
38. The Babri Masjid was a mosque that was demolished in a case that will be discussed later in this article. The demolition also triggered large bouts of communal violence. The term 'communal violence', however, often obscures the superior position of Hindu rioters in government and business, and their concomitant advantages in 'violence'.
39. Jaffrelot, loc. cit.
40. Ibid., p. 3.
41. Sheela Bhatt. "Rediff.Com: Vajpayee Lashes out at Muslims". In: *Rediff* (Apr. 12, 2002). URL: http://www.rediff.com/news/2002/apr/12bhatt.htm (visited on 08/31/2018).
42. "Timeline: Ayodhya Holy Site Crisis". In: *BBC News. South Asia* (Dec. 6, 2012). URL: https://www.bbc.com/news/world-south-asia-11436552 (visited on 09/01/2018).
43. 'Naxal' is a term for some Maoists.

What causes in this instance nostalgia for Vajpayee is a broader tendency to romanticise the past. This is why liberals dream of Nehruvian times, Hindu nationalists dream of the time before the arrival of the Muslims, and regional movements dream of their zeniths. Such nostalgia is natural in a polity whose every moment has been so profoundly unsettling. Partition most likely killed between two hundred thousand and two million.[44] There are countless cases of abuse of authority: civil servants, police officers and soldiers rape, torture and abduct with impunity. Six years ago, 3,000 died per day of hunger.[45] The situation at independence must have been even worse. When more than six decades after liberation from British imperialism the state still failed to prevent deaths from starvation it is difficult to have hope that these fundamental problems will be resolved. Thus a past is selectively constructed of carefully selected elements—Nehru's Anglophilia, the genius of Vedic civilisation, or the genesis of the Tamil language, and comes to dominate particular political strata.

It is also appropriate, therefore, to reëvaluate the idea that India presently suffers from a unique anti-democratic malaise. To suggest that the political situation at present of India is unique is profoundly ahistorical. Post-1947 India has known, and almost certainly will continue to know political hegemony. The most obvious objection is in the very vocabulary of those who decry the present situation: it is 'a new Emergency', 'as bad as Emergency', a 'reminder' thereof, and so on.

India today still rests on the administrative framework of the Raj. This is apparent in the Indian Administrative Service, which operates on the same basis as the Imperial Civil Service that preceded it. Just as the post-nominal letters I.C.S. stood for the social system that produced the men who staffed it, the balance of force that ensured that those men could wield absolute authority, and the constitutional absurdities that allowed them to proceed in good conscience, so too do the letters IAS, save that there are more quotas and women. The same army intervenes in troubled areas to keep down rebellion. The same police lathi charge trade unionists in the cities, and even urban liberals protesting the lack of state action to halt rape, as seen in 2012. The change is that the lathis are occasionally replaced with water cannons, a few more lines of Kipling's poem now refer to brown and not white men, and that such action now can be prosecuted without offending the consciences of the bourgeoisie, for they are done in the name of an 'Indian' instead of 'foreign' state. The same Indian Penal Code, containing provisions against 'sedition', 'blasphemy' and homosexuality (under the infamous § 377), applies. British justice is inherited in the form of preventative detention.

Under Nehru a personal devotion to the family was created that paved the way for Emergency. Even *habeas corpus* rights were suspended.[46] Regionalism has largely given way to dictatorship—political violence in West Bengal[47], Kerala[48], Tamil Nadu[49] and so on mirrors arrests orchestrated by the BJP.

The complicity of the Congress Party in violence against Indian Sikhs after the assassination of Indira Gandhi is also well-documented.[50] What is now forgot is the coercive manner in which the Union of India whose politics it came to dominate for perhaps five cumulative decades was established—the process of Hyderabad's accession[51], for example, shows that a dependence on coercion continued till after independence.

Vajpayee was neither much better nor much worse than other Indian politicians. As a person, he may well have mellowed on his journey from Ahodhya to 7, Race Course Road. What he represents is a fundamentally empty ideology, saved only by a 'pragmatism' that managed to embed a little normative content capable of preventing a total slide into a Hindutva hell. The nostalgia that presently causes his veneration will probably be of little import—Modi et al. are capable of drawing crowds too without resorting to appeals to a previous generation of leaders. His death was regrettable, but was most likely a good one. He was not, for example, assassinated. At his funeral he was surrounded by Chief Ministers and a Prime Minister from his own party. The India he departs increasingly meets the demands he made of it—militarily and commercially it strengthens not so much because of any deliberate policy as because of a progression engendered by the removal of Nehruvian restrictions. He died peacefully in an Indian hospital, treated by Indian doctors who quite possibly attended Indian universities.

That fundamentally empty ideology, however, seems to promote collective decision-making in a manner that the Congress Party never will so long as it remains dependent on the Nehru-Gandhis. Vajpayee, as adept in coalition as in navigating the currents of the Sangh Parivar's internal politics, perhaps made the single largest individual contribution to its political ascendancy. This loss of fallibility can only be to the cost of the victims of the Indian state, whose only salvation is often its fast-disappearing incompetence; by rendering the BJP a viable national political force, it may also be Vajpayee's principal legacy. There may, however, come a day when the Indian state becomes capable of serving its citizens more than by accident—then, perhaps, the iron frame may become of use.

44. Ian Talbot. "Partition of India: The Human Dimension: Introduction". In: *Cultural and Social History* 6.4 (Dec. 2009), pp. 403–410. ISSN: 1478-0038, 1478-0046. DOI: 10.2752/147800409X466254. URL: https://www.tandfonline.com/doi/full/10.2752/147800409X466254 (visited on 09/03/2018).
45. Mayank Bhardwaj. "As Crops Rot, Millions Go Hungry in India". In: *Reuters* (July 1, 2012). URL: https://www.reuters.com/article/us-india-wheat/as-crops-rot-millions-go-hungry-in-india-idUSBRE8600KD20120701 (visited on 09/03/2018).
46. A.N. Ray. *Additional District Magistrate, Jabalpur vs. S. S. Shukla*. Apr. 28, 2018. URL: https://indiankanoon.org/doc/1735815/.
47. Shoaib Daniyal. "What Makes the Politics of West Bengal so Violent?" In: *Scroll.in* (July 5, 2018). URL: https://scroll.in/article/881357/what-makes-the-politics-of-west-bengal-so-violent (visited on 09/03/2018).
48. Apoorvanand. "The Onus Is on CPI(M) To Put an End to Competitive Political Violence in Kerala". In: *The Wire* (Feb. 15, 2018). URL: https://thewire.in/politics/the-onus-is-on-cpim-to-put-an-end-to-competitive-political-violence-in-kerala (visited on 09/03/2018).
49. Bhanupriya Rao. "Tamil Nadu: How Violence Is Used as the Ultimate Tool of Intimidation against Assertive Women Leaders in Politics". In: *Firstpost* (Aug. 21, 2018). URL: https://www.firstpost.com/india/tamil-nadu-how-violence-is-used-as-the-ultimate-tool-of-intimidation-against-assertive-women-leaders-in-politics-4440561.html (visited on 09/03/2018).
50. "Delhi 1984: Memories of a Massacre". In: (Nov. 1, 2009). URL: http://news.bbc.co.uk/1/hi/world/south_asia/8306420.stm (visited on 09/03/2018).
51. Taylor C. Sherman. "The Integration of the Princely State of Hyderabad and the Making of the Postcolonial State in India, 1948–56". In: *The Indian Economic & Social History Review* 44.4 (Dec. 2007), pp. 489–516. ISSN: 0019-4646, 0973-0893. DOI: 10.1177/001946460704400404. URL: http://journals.sagepub.com/doi/10.1177/001946460704400404 (visited on 09/03/2018).

Adventures in Recreational Mathematics VII: Numbers that are not astronomical in size

Isky Mathews

Salutations!

This time we will examine some of mathematics' largest numbers; I feel this is a conceptual area that many find interesting, and even amusing, to think about—some of the numbers I will mention here will be so large that to refer to them as *astronomical* would not just be inaccurate, since there is no object that could found in these quantities within the observable universe, but, frankly, *insulting* to their magnitude.

To demonstrate the previous point, we shall begin by considering the number of baryons in the observable universe. Baryons are particles made up of 3 quarks and interact with the strong nuclear force, e.g. protons or neutrons, and we can calculate how many there are using 4 numbers, 3 of which were obtained using data from the Planck Satellite:

- ρ_{crit}, the critical density of the universe ($= 8.64 \times 10^{-33} kgm^{-3}$)
- Ω_b, the fraction of the universe's energy in baryons ($= 0.0485$)
- L, the radius of the observable universe, which is roughly spherical ($= 4.39 \times 10^{26} cm$)
- m_p, the mass of one proton ($= 1.67 \times 10^{-27} kg$)

Now, as ρ_{crit} is essentially the energy density of the universe, $\rho_{crit} \times \Omega_b$ is the mass stored in baryons per cm^3 of the observable universe on average, making $\rho_{crit} \times \Omega_b \times \frac{4}{3}\pi L^3$ roughly the combined mass of all baryons in the universe. Finally, because a neutron's mass is essentially equivalent to that of a proton, we divide the above expression by m_p to get

$$\frac{\rho_{crit} \times \Omega_b \times \frac{4}{3}\pi L^3}{m_p} = 8.89 \times 10^{79}$$

which is really quite a big number, in comparison to the numbers of things you encounter in everyday life. However, it was small enough to be expressed, to a fair level of precision and concisely, using a notation with which we are so familiar that I barely need to name it: that of the *exponential*. For many, if asked to write down quickly the biggest number they could think of at the time, exponentials or stacked exponentials of the form

$$a^{b^{c^{d^{\cdots}}}}$$

would be their first thought, due to its simplicity—for example, just 10^{10^2} is more than the number of baryons in the universe. In fact, our first famous number can be expressed as 10^{100}, a *googol*, and the next as $10^{10^{100}}$, a *googolplex*. We shall return to exponentials and the process of stacking them later, for it has great potential to make large numbers.

Primitive Recursive and Non-Primitive Recursive functions

For now, we take ourselves back to near the beginning of the 20th century, when individuals such as **Gödel, Turing** and **Church** were discussing the nature of functions. They realised that the process of calculating the outputs to most functions could be seen as an iterative process that, most importantly, had a predictable number of steps; for example, to calculate $2 + 2$, one could see it as applying $f(n) = n + 1$ to the input 2 twice. Such functions were called *primitive recursive*, because they *could* be written down or represented recursively, i.e. where they were seen as a series of repeated applications of some function, but could also be written down in a single closed form—all polynomials, exponentials and many more that we are familiar with are primitive recursive. The computer scientist Robert Ackermann is most famous for describing an eponymous function, denoted $A(m, n)$, that was still possible to evaluate but was not primitive recursive, defined by these conditions:

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

Let us call a *closed-form* representation of a function a form which uses a finite number of operations and without self reference. Then, an amazing fact is that the Ackermann function's above self-referential or *recursive* definition cannot be written out into a closed form, unlike addition or multiplication—this means it is not a primitive-recursive function and it grows extremely quickly—try evaluating it for different inputs! Clearly things like $A(0, 3) = 4$ and $A(1, 2) = 4$ are quite small, but then $A(4, 3)$ is an incredible 19729 digit number:

$$A(4,3) = 2^{2^{65536}} - 3$$

In fact, it's often difficult to find examples to demonstrate how large the numbers that the Ackermann function outputs are, because nearly all of them are so big that they either can't be written down in any concise manner or, worse, they couldn't be computed within the lifetime of the universe given all the computing available today. Furthermore, Ackermann and his peers were later able to show that functions of this kind[0] *dominate* all primitive recursive functions, i.e. for any primitive recursive function $f(x)$ and a non-primitive-recursive function $g(x)$, there is some input n so that for all $m > n$, $g(m) > f(m)$.

In order to understand and express just *how* quickly such functions grow, we have to use a lovely notation developed some years ago by the famous **Donald Knuth**[1] known as *up-arrow notation*, which is based on the idea of the *hyperoperation hierarchy*. The first operator in the hierarchy is the *successor*, an unary operator (meaning that it takes 1 argument) which takes in n and outputs $n + 1$, often written $n++$.[2] Addition can be seen as repeated successorship in that $a + b$ can be seen as denoting $\ldots((a\ +\ +)\ +\ +)\ +\ +\ldots$, where the successor operation is applied b times. Similarly, multiplication

0. As in, those that can be evaluated in a finite amount of time but that are not primitive recursive.
1. A computer scientist and mathematician, perhaps most famous for his remarkably complicated series of volumes *The Art of Computer Programming* (often referred to as the computer scientist's bible!) but also for the typesetting system TEX, whose offspring, LATEX 2_ε, this very publication uses to format its articles!
2. This is, interestingly, why C++ is called what it is—it was supposed to be the *successor to C*

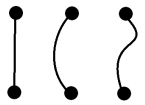


Figure 0: These three 2-point graphs are the same.

is repeated addition, as $a \cdot b$ is equal to $a + a \ldots + a$ where a appears b times. We can make a hierarchy of such functions, with $a *_1 b = a \cdot b$ by definition, and thereafter, $a *_n b$ defined as $a *_{n-1} (a*)$ with b instances of a in that definition. Knuth created the hyperoperation-notation $a \uparrow b$ which *starts* at exponentiation (as in $a \uparrow b = a^b$) and by writing more arrows, one goes up the hierarchy, so $2 \uparrow\uparrow 4 = 2^{2^{2^2}}$ —the name we give for this operation above exponentiation is "tetration" and $a \uparrow\uparrow\uparrow b$ is called "a *pentated* by b" etc. These operations make writing really large numbers simple and if we *index* the arrows, that is say that $\uparrow^n$ denotes n arrows, then we can write down numbers that could never have any practical use—for example, the famous **Graham's number**.

Graham's Number

This number comes out of a question in a somewhat ill-defined area of mathematics known as Ramsey Theory, which purports to comprehend the conditions under which complex structures are forced to appear; **Ronald Graham** and **Bruce Lee Rothschild**, both legends in this field, came up with the question in 1970. The question requires understanding what a *graph* is in pure mathematics; Benedict Randall Shaw has written a helpful article explaining graph theory in a previous issue of *The Librarian*[3], but a summary is that any set of points and lines drawn connecting them is a graph. More formally, a graph is a set of points along with a set of pairings defining connections between those points—thus neither the precise coordinate/relative position of points nor the shape of the lines connecting them matters, only the connections[4].

Given n points, the graph obtained by adding all possible connections between them is called the *complete graph on n vertices*, denoted K_n (e.g. K_3 is like a triangle and K_4 is like a square with its diagonals drawn in). Now, Rothschild and Graham were considering complete graphs on n-dimensional cubes[5], which have 2^n vertices each, and properties of the *colourings* of their edges, i.e. the ways in which you can assign different colours to those edges. In particular, they asked what was the smallest value of n such that every 2-colour colouring, using, for example, red and blue, of the edges of the complete graph on an n-dimensional cube is *forced* to contain a subset S containing exactly 4 of its points such that all the edges between the points in S are the same colour and such that all points in S are *coplanar*[6]. They were able to prove that there is such an n, and they knew from checking on paper that $n > 5$, and so they sought to also put an upper-bound on it (Graham's number)[7]. It is constructed as follows:

- Let $G_1 = 3 \uparrow^4 3$ (an amazingly large number, so big that the number of 3s in its power-tower representation couldn't be written in base 10 even if each digit could be assigned to each planck-volume in the observable universe!)
- For each n, let $G_{n+1} = 3 \uparrow^{G_n} 3$
- Then Graham's number is G_{64}.

It is clear from this that uparrow notation becomes inadequate for integers as large as Graham's Number, since there is no way of expressing it concisely if we need to write out all the arrows. Thus, when you have gotten over G_{64}, we must move on to a better framework that will allow us to see just how large it is "in the grand scheme of things".

The Fast-Growing Hierarchy *or* the Grandest of Schemes of Things

The fast-growing hierarchy is a series of functions, built recursively, that grow faster and faster as we go up. We start with the simple function $f_0(x) := x + 1$ and we say[8] that $f_1(x) := f_0^x(x)$, or in other words $x + x$. Similarly, $f_2(x) := f_1^x(x) = x \times x$ and in general for any integer $n > 0$, $f_n(x) = f_{n-1}^x(x)$.

So far, there is no difference between this and hyperoperations but now, we can use *ordinals* to give us unbounded growth-rates... There was a previous article[9] introducing readers to the wonderful universe of ordinals but, to simplify their technical definition, they are a clever set-theoretic version of numbers, discovered by **Georg Cantor**, which essentially allows us to have a natural extension of the integers to varying sizes of infinity. The number ω is the ordinal "larger" than all the integers but then we still have a well-defined concept of $\omega + 1$ or $+2$ or $+n$ and much, much more. We call ω the first *limit ordinal*, meaning that it has no specific predecessor, but rather can be reached as a limit of a strictly increasing sequence, and we call $2, 3, 4, n, \ldots$ and $\omega+1, \omega+2, \omega+n$ etc. *successor ordinals* because they *do* have a well-defined predecessor (i.e. they are the successor of some known ordinal). Thus we have the definition that if α is a successor ordinal, then $f_\alpha(x) = f_{\alpha-1}^x(x)$, and if α is a limit ordinal and S_α is a strictly-increasing sequence of ordinals whose limit is α (as in, α is the smallest upper-bound for all the terms in S_α), with $S_\alpha[n]$ denoting the nth term of S_α for some ordinal n, then $f_\alpha(x) = f_{S_\alpha[x]}(x)$.

To give an example[10], $f_\omega(x) = f_x(x)$, since the sequence of integers 1,2,3,...,x,... has the limit ω but since $\omega + 1$ is a successor ordinal, $f_{\omega+1} = f_\omega^x(x)$. We can observe from these definitions immediately that $f_\omega(x)$ can't be primitive-recursive, since it grows faster than any f_n for integer n, and thus that it is, in a sense, *beyond uparrows*, since it can't be represented in the form $m \uparrow^k x$, where m, k are fixed integers. In fact, it is possible to show that $f_\omega(x)$ grows at almost exactly the same

3. Benedict Randall Shaw. "An Introduction to Graph Theory". In: *The Librarian Supplement* 1.2 (Nov. 7, 2017). URL: `https://librarian.cf/s2v1/graphtheory.html`.
4. To be precise, we consider two graphs that have the same number of vertices and the same connections between those vertices but are drawn differently to be distinct graphs or objects but we say they are *isomorphic*, i.e. share all the same graph-theoretic properties.
5. Benedict Randall Shaw, the mathematic editor, produced a diagram of such an hypercube in four dimensions that was reproduced on the front cover of a previous issue.
6. i.e. are points on a common plane.
7. It may be of interest that subsequently we have created a better bound, $2 \uparrow\uparrow\uparrow 6$.
8. Here, $g^n(x)$, for some integer n and some function $g(x)$, denotes performing g to the input x, n times.
9. Isky Mathews. "Adventures in Recreational Mathematics V: Cantor's Attic". In: *The Librarian Supplement* 1.1 (Oct. 9, 2017). URL: `https://librarian.cf/`.
10. Some may notice that this definition only applies for integer x (since there is no 3.2th function in our list, for example)—that's because of the caveat that the fast-growing hiearchy only contains functions defined for ordinal inputs.

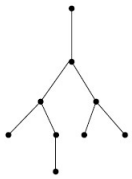


Figure 1: A tree.

rate as the Ackermann function that we've seen previously and that $f_{\omega+1}(64) > G_{64}$.[11] Now, you can choose your favourite transfinite ordinal and create a function that grows faster than you can imagine, for example $f_{\omega\times 2}$, f_{ω^2}, f_{ω^ω} or, if $\epsilon_0 = \omega^{\omega^{\omega^{\omega^{\cdots}}}}$, then you can even have f_{ϵ_0} and larger!

Kruskal's Tree Theorem and $TREE(3)$

Kruskal's Tree Theorem, conjectured by Andrew Vazsonyi and proved in 1960 by Joseph Kruskal (an influential combinatoricist), is a statement, once again, relating to graphs and to explain it, we need some more vocabulary concerning them.

We say that, given a graph G and a point p in G, if there is a way of starting at p and traversing a finite number of edges (that is greater than 2) to move through a sequence of distinct vertices of G which eventually ends up at p again, then we call such a path a *cycle* and we call a graph *acyclic* if it doesn't doesn't contain any cycles. *Connected graphs* are what they sound like—they are graphs with the property that for any two of its vertices, there is always a path between them. Further, we say that a connected, acyclic graph G that is also *rooted*, i.e. there is one vertex we call the *root* and every other vertex is considered (and often drawn) "below" that root, is called a *tree*. This makes sense intuitively and Fig. 1 gives an example of a tree to demonstrate. Similarly, peiople call the vertices at the ends of branches of trees *leaves* and, if one vertex v_1 is closer to the root than v_2 and their paths to the root overlap, then we call v_1 a *parent* of v_2.

A k-labeled tree is one where each of the tree's vertices are assigned one of k "labels", which for our purposes we may consider as colours. The most complicated definition that Kruskal's Theorem requires us to consider is the idea of a k-labeled tree T_1 being *homeomorphically embeddable* (h.e. from now on) into another k-labeled tree T_2. Given T_1 and T_2, we say that T_1 can be h.e. into T_2 if there is a function $p(x)$ that takes as inputs vertices of T_1 and outputs vertices of T_2 with the properties that

- for every vertex v from T_1, v and $p(v)$ have the same colour;
- for every pair of vertices v_1, v_2 from T_1 and if $xANCy$ denotes the closest common parent of two vertices x and y, then $p(v_1 ANC v_2)$ has the same colour as $p(v_1)ANCp(V_2)$.

To illustrate this somewhat abstract definition, you can look at Fig. 3 which gives an example of two k-labeled trees with the first being h.e. into the other. Finally, Kruskal's Tree Theorem states that for any k and for any infinite sequence of k-labelled trees $T_1, T_2, T_3, ...$, where each T_n can have at most n vertices, it is true that for some i and some $j > i$, the tree T_i is h.e. into T_j. While the theorem itself doesn't allow us to describe large numbers, the mathematician Harvey Friedman

11. They aren't actually comparable in size, since $f_{\omega+1}(64) > f^{64}_{\omega}(6) > G_{64}$.

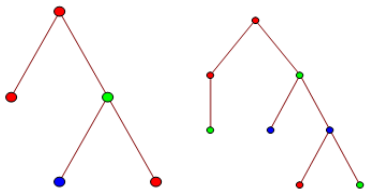


Figure 2: The left 3-labeled tree is h.e. into the right 3-labeled tree.

made the observation that the theorem allows us to define the function $TREE(n)$, which returns the length of the *longest finite sequence* of n-labeled trees such that no T_i is h.e. into a T_j for integers i and $j > i$ and it turns out that k-labellings of trees are amazingly combinatorially rich:

- $TREE(1) = 1$, since T_1 can have at most 1 vertex and at most use 1 colour, so it is simply a single coloured point and that is clearly h.e. into any successive trees of the same colour.
- $TREE(2) = 3$, where the sequence begins with a single vertex of Colour 1, then we have a two-vertex tree of Colour 2 and then we have a single vertex of Colour 2—can you see why this is the longest such sequence?
- $TREE(3)$ then is so vastly, incredibly large that I struggle to find description for it.

The first thing we can say is that an extremely lower bound that can be proven for it is $f_{\omega}^{f_\omega(187196)}(2)$, which is clearly immensely bigger than $f_{\omega+1}(64) > G_64$! I shall now attempt to explain how far up one needs to go in the fast-growing hierarchy to reach a function that can rival $TREE(n)$.

In the 20th century, the mathematician Veblen, along with others, was attempting to create a schema for notating and comparing really large infinite ordinals; he first proved that if one has a function $h(x)$ that takes as inputs and outputs ordinals, that is strictly increasing, and whose value for a limit ordinal λ is equivalent to the limit of the sequence $h(a_0), h(a_1), h(a_2), ...$ where the limit of $a_0, a_1, a_2, ...$ is λ,[12] then $h(x)$ has fixed points for some ordinals. This remarkable property of the ordinals is part of what makes them so interesting to study for set-theorists—they seem to be "averse" to notation systems for them. Regardless, Veblen created his *Veblen hierarchy*, a series of functions $\phi_0(x), \phi_1(x), \phi_2(x), ...$ where $\phi_0(x) = \omega^x$ and each $\phi_{n+1}(x)$ is equal to the xth fixed point of $\phi_n(x)$—take $\phi_1(1)$, for example, which is the first fixed point of ω^x, i.e. ϵ_0 as discussed previously. Now, the *Feferman-Schütte ordinal* Γ_0 is the smallest ordinal α that satisfies the impressive $\phi_\alpha(0) = \alpha$ or, to paraphrase Solomon Feferman himself, Γ_0 is the smallest ordinal that cannot be reached by starting with 0 and repeatedly using addition and the Veblen hierarchy of functions[13] (in fact, it is very difficult to create notation systems that can describe ordinals above Γ_0 as well as all those below it).

What we can now explain is that despite the magnitude of Γ_0, it is possible to show that the growth rate of $TREE(n)$ is much greater than that of $f_{\Gamma_0}(x)$. But, ultimately, $TREE(n)$ is piffle

12. Such a function is commonly called a *normal function* in set theory.
13. Kurt Schütte. *Proof Theory*. Red. by S. S. Chern et al. Vol. 225. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg: Springer Berlin Heidelberg, 1977. ISBN: 978-3-642-66475-5. DOI: 10.1007/978-3-642-66473-1. URL: http://link.springer.com/10.1007/978-3-642-66473-1 (visited on 08/29/2018).

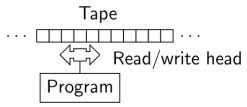


Figure 3: An illustration of a Turing machine.

in comparison to what comes next—after all, it grows slow enough for it to still be computable...

Computability and busy beaver numbers

You recall the distinction between recursive and primitive-recursive functions made by those great minds near the beginning of the 20th century? That result is but one in a bigger theory of computation called, aptly, computability theory. Computability theorists study fundamentally what it means to compute something by examining models of computation.

Mathematicians in this era were attempting to go back to the foundations of mathematics and to *formalise* them, that is to make them completely logically rigorous and unambiguous—this was because it had been discovered that without such careful thought and by relying on implicit or vague definitions, one can run into paradoxes or questions that don't have an answer, simply because they don't mean anything. One of the interesting notions that Church and Turing picked up on this journey was that most people appeared to assume that all functions and questions in mathematics were answerable and simply required "proper definitions" and a great amount of thought—their great insight was that the process of logically working things out or deducing true statements had mathematical properties in and of itself and thus realised that it was possible to create problems that were well defined (as in, they had a unique answer) but were beyond the capability of any proof-system or computational device to solve; such problems are known as *uncomputable problems*.

Computability theory is such an amazing area of mathematics that it deserves its own article[14] but we shall simplify here to explain just the concepts required for describing large numbers. Alan Turing came up with the concept of *Turing machine* (TMs from now on), which are theoretical automata that have a *tape* that stretches out infinitely in one direction and is divided up into discrete tape-segments, along with a *tape-head* that is capable of reading and writing symbols on the tape.

It also has a number of *states* that it can be in and a rulebook that determines, given that the TM is in the nth state and that the symbol being read by the tape-head is X, what symbol Y the TM should overwrite X with and whether to then move one tape-segment right or left. It should finally be noted that a TM has two special states denoted YES or NO which, when reached during some stage of a TM-computation, cause the TM to *halt*, i.e. cease all movement. Turing and Church believed that a Turing machine should be able to perform any algorithm that is well-defined because:

- They (and other mathematicians such as Gödel) had previously created other systems which were supposed to represent computation, such as the λ-*calculus* and the set of general recursive functions, and it turned out that the set of problems they could solve were all equivalent.
- They were able to show ways of creating Turing machines that could evaluate many well known algorithms, such as one that could perform primality-checks or could multiply two numbers together.
- They lived to see John von Neumann, a startingly brilliant mind, design modern-day computer architectures using their ideas and create the first computers.

In fact, in the 1960s and 70s, many individuals tried to create other deterministic systems that they believed might be able to compute more than Turing machines but all were eventually proved to be able to solve the same set of problems—thus, they proposed the *Church-Turing thesis*: that every sequence of logical steps that were performable by a human would also be performable, theoretically, by a Turing machine, and vice versa; subsequently, any system that was capable of solving the same set of problems as TMs came to be known as *Turing-complete*.

However, as I mentioned previously, those problems Turing-complete systems can solve do not encompass *all* problems, they only contain *computable* problems. But what, then, would an uncomputable problem look like? Well, Turing made the observation that certain Turing machines are setup such that, after some finite number of steps, they halt in the state YES or NO and others never halt and simply continue moving along their tape indefinitely. In 1936, he published a paper describing the *Halting problem*'s uncomputability: it is impossible to create a *general, finite-time algorithm* to decide accurately whether a given TM halts—he showed this through a clever proof-by-contradiction. To illustrate it, we first remember that, by Turing-completeness, any computability theorem that applies to Turing machines applies to many of the modern-day computer programming languages[15], so we shall precede by writing our proof in the language of *Python*. Let us assume, hoping to reach a contradiction, that there was a program defining a function called *Halts(x)*—this takes in the name of a programmed-function for which you wish to check whether it halts or not and outputs after a finite amount of time either *True* or *False* as per the answer. Then consider the function:

```
def contradiction(x):
        if Halts(contradiction(x)) == True:
                while True:
                        x = x
        else:
                return x
```

For those less familiar with Python, *contradiction(x)* is designed to halt after a finite amount of time if and only if *Halts(x)* says that it will run forever and *contradiction(x)* will loop forever if and only if *Halts(x)* says that it will halt after a finite amount of time. This function is clearly contradictory and so our initial assumption, that there could exist such a function *Halts(x)*, is false. Thus, the Halting problem is uncomputable!

An interesting observation, then, made by a researcher at Bell Labs was that, for a given n, it *must be true* that there is some number k representing the largest number of steps that a *halting* n-state Turing machine (i.e. a TM which does eventually terminate its computation) will take before stopping—the researcher called these TMs *Busy Beavers* and the corresponding

14. Perhaps a series of articles!

15. If we assume they are being run on a computer of unlimited memory and that they are capable of performing genuine arbitrary-precision arithmetic.

k for each n the nth *Busy Beaver number* ($BB(n)$), because they take a long time to stop and the way that they move up and down their tape during their computation reminded him of a beaver making a dam. So, by definition, there is no n-state Turing machine that takes longer than $BB(n)$ steps to halt.

However, he pointed out that there cannot exist a general algorithm to compute $BB(n)$ for each input n because if there was one, then we would have the following finite-time algorithm to solve the Halting problem:

- If the TM for which we are trying to determine whether it halts or not has n states, compute $BB(n)$.
- Then, start running the TM and wait $BB(n)$ steps. If the TM halts by this time, then we know that it halts. If the TM does not halt by this time, then we know, by the definition of $BB(n)$, that it does not halt.

but such an algorithm cannot exist by the uncomputability of the Halting problem and so $BB(n)$ is an uncomputable function. Further, there cannot be any computable function $g(n)$ which acts as an upper-bound to $BB(n)$ for each n, because if that was true then similarly we could wait $g(n)$ steps to determine whether an n-state TM halts (any TM which takes longer than $g(n)$ steps to halt must have already taken more than $BB(n)$ steps, and so cannot halt), giving us again an impossible finite-time algorithm to solve the Halting problem. So, $BB(n)$ isn't just uncomputable—no computable functions can act as an upper-bound to it. So all the functions we have seen previously—$A(m,n), f_\omega(n), f_{\epsilon_0}(n), f_{\Gamma_0}(n), TREE(n)$—they all must inherently grow slower than $BB(n)$ simply by the fact that they are computable.

Mathematicians have worked out that $BB(0) = 0, BB(1) = 1, BB(2) = 4, BB(3) = 6$ and that $BB(13)$ simply by trying and examining all n-state Turing machines for $n < 5$ but only lower bounds are known beyond this—for example, we know that

$$BB(7) > 10^{10^{10^{10^{18705353}}}}$$

and that

$$(BB(5) > 3^{3^{3^{3^{3^{\cdots}}}}}$$

where the number of threes is 7625597484987.

But now we can go further even than this to functions and numbers so fantastic that I will at the end write down a number that, to my current knowledge, is greater than anything ever written in any mathematics publication, formal or informal as it is something of my own design. Because computability theory is all about imagining theoretical scenarios related to computation that could never happen—for example, we could never build an actual TM, since it requires an infinitely long tape—one thing we can consider is an *oracle*, a purely theoretical construct that would allow us to compute certain uncomputable functions. For example, we could have the oracle $BB_0(n)$, which outputs the nth Busy Beaver number, although such a thing could never normally exist in our universe even if it was infinite but remember, this is *pure mathematics*—oracles can exist because we say so and they have interesting theoretical properties. For example, if you take the set S_0 of TMs equipped each with $BB_0(n)$, the analogous version of the halting problem for S_0 is still uncomputable by those TMs in S_0 and so S_0 has its own associated Busy Beaver function, $BB_1(n)$, that grows faster than any function computable by S_0 TMs (and thus grows far faster than $BB_0(n)$). Similarly, we could then introduce an oracle for $BB_1(n)$ and create a new set of Turing machines S_1 equipped with both the $BB_1(n), BB_0(n)$ oracles and, once again, S_1 would have its own version of the Halting Problem and thus a well-defined, but uncomputable function $BB_2(n)$ that grows even faster than $BB_1(n)$.

By iteratively creating more and more sets $S_0, S_1, S_2, ...$, we get faster and faster growing functions assigned to them $B_0(n), B_1(n), B_2(n), ...$ and so, as with the fast-growing hierarchy, we can allow for *ordinal* subscripts where if λ is a limit ordinal, then $B_\lambda(n)$ is the Busy Beaver function for Turing machines with oracles for all ordinals below it. Thus, we can have $B_\omega(n)$ and $B_{\epsilon_0}(n)$ etc. Now, we call an ordinal α *computable* if there is a Turing machine which can tell for any two distinct ordinals β, γ smaller than α whether $\beta > \gamma$ or $\gamma > \beta$ (for example, ω is a computable ordinal, since any good computer can tell, given two nonequal integers, which one is bigger than the other). So, I define the *Ultra-function* $U(n)$ to be $B_{\delta(n)}(n)$, where $\delta(n)$ is the largest ordinal computable for n-state Turing machines - so $U(n)$ grows faster than any individual $B_\alpha(n)$!

Thus, as a final answer to *Who can name the bigger number?*, I advise you to simply write down $K_\Omega = U(TREE(TREE(TREE(3))))$, which is quite concise but which I can verify, thanks to some results on much smaller numbers, is so big that if your opponent attempts to describe another number N, whether the statement $N > K$ is true is in fact *independent* of ZFC-set-theory (i.e. cannot be *proved or disproved* using most modern-day mathematical techniques, regardless of whether it seems obvious).

If you enjoyed this article and would interested in reading more, here are some titles of subjects and numbers that you can look up for further reading:

- Rayo's Number (before I wrote K_Ω's definition above, it *was* the biggest number ever written down in human history)
- Chaitin's constant, the Halting Problem and other uncomputable problems.
- Ordinals and Cardinals in Set Theory.
- Ramsey Theory and Ramsey Numbers.
- Go to the Googology Wiki, a fun website dedicated to listing and describing some of the largest numbers mathematicians know of!

And now for the challenges. For those who haven't read ARM before, each article ends with 2 challenges and if you get a solution to either one, please email either **Isky Mathews** (isky.mathews@westminster.org.uk) or **Benedict Randall Shaw** (benedict.randallshaw@westminster.org.uk):

Challenge 1: Determine whether $10^6 \uparrow\uparrow 10^6 > 3 \uparrow\uparrow\uparrow 3$ or $3 \uparrow\uparrow\uparrow 3 > 10^6 \uparrow\uparrow 10^6$.

Challenge 2: Can you think of a way of describing bigger numbers than using the "uncomputability hierarchy" above along with some form of (minimally-heuristic) argument as to why you believe they are bigger? (This would be exceedingly impressive, not just to us but to the entire mathematics community.)

On the International Mathematical Olympiad

Benedict Randall Shaw

The International Mathematical Olympiad consists of two four-and-a-half hour exams, with six problems in total; solutions to the first five are presented here, with occasional comment on how one might find them.

Problem 1

Let Γ be the circumcircle of acute-angled triangle ABC. Points D and E lie on segments AB and AC, respectively, such that $AD = AE$. The perpendicular bisectors of BD and CE intersect the minor arcs AB and AC of Γ at points F and G, respectively. Prove that the lines DE and FG are parallel (or are the same line).

(The circumcircle of triangle ABC is the circle that passes through its three vertices; the perpendicular bisector of a line segment is the line that is perpendicular to it and passes through its midpoint.)

It is quite useful to draw this diagram for oneself; we recommend that readers construct this diagram with a compass and straight edge, and make additions as we progress through the proof, as an aid to understanding; but failing that, we have provided computer-drawn diagrams.

There were a variety of ways to solve this geometry problem; in fact, each member of the British team boasted a different one. This is the one the author presented in the competition.

We are asked to find that FG is parallel to DE; it would be helpful to have a better characterisation of this property. $AD = AE$ does look rather symmetric about the bisector of $\angle DAE$. Let J be the midpoint of DE.

Claim 1. AJ is perpendicular to DE, and the bisector of $\angle DAE$.

Proof. We know $AD = AE$ (given in the question), $DJ = JE$ (by the definition of J), and $\angle EDA = \angle AED$ (as triangle ADE is isosceles); so by SAS congruency, triangles ADJ,AEJ are congruent.

Thus $\angle DAJ = \angle JAE$, so AJ is the bisector of $\angle DAE$; but also $\angle AJD = \angle EJA = 90°$, so DE is perpendicular to AJ.

We want to show that FG is parallel to DE; it would therefore suffice to show that FG was also perpendicular to the bisector of $\angle BAC$, as then FG and DE, both being perpendicular to this common line, would be parallel.

Since we have the perpendicular bisectors of BD, CE already, it would seem fairly natural to introduce the midpoints of these segments. Let K, L be the midpoints of BD, CE respectively.

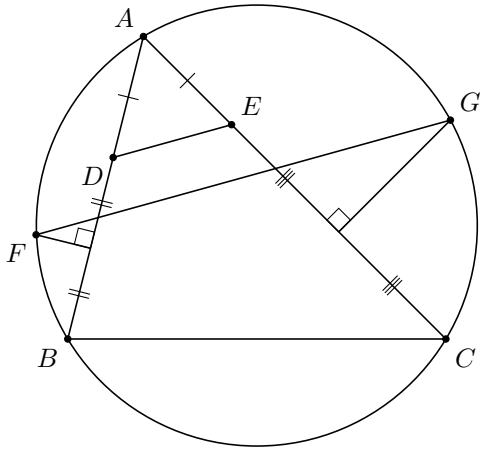


Figure 0: The configuration, as given in the question.

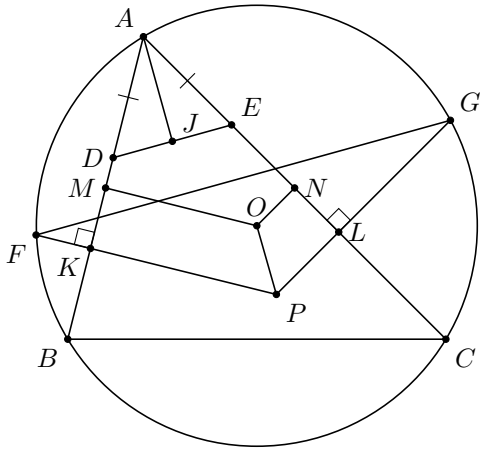


Figure 1: The configuration, with some additions.

It is a well-known fact[o] that the perpendicular bisectors of AB,AC coincide at the centre of Γ. Since we are dealing in perpendicular bisectors of BD, CE already, we may as well add these other bisectors; and in fact, when one constructs the diagram by hand, one may draw them in just to find the centre of Γ. Let the centre of Γ be O. Let the midpoints of AB, AC be M, N respectively. Then OM, ON are the perpendicular bisectors of AB, AC respectively.

The lines FK, GL look a bit like OM, ON, in that they're similarly defined (being perpendicular bisectors), and are respectively parallel. Maybe their intersection is also of interest? Let P be the intersection of FK, GL. Since O and P seem similarly defined, let's mark in OP on the diagram.

Here the keen-eyed reader will spot that AJ looks suspiciously parallel to OP. How would we prove this? If we could show that some vector parallel to AJ takes line MO to KP, and NO to LP, then that vector would take O, the intersection of MO, NO, to P, the intersection of KP, LP. This vector is clearly $\overrightarrow{OP}$.

We want to show this vector takes MO to KP, and NO to LP. Since we already know MO, KP are parallel, as are NO, LP, it suffices to show that this vector takes M to a point on KP, and N to a point on LP. What are these points? Let Q be the point on FP such that MQ is parallel to AJ, and R the point on GP such that NR is also parallel to AJ.

o. If you have no prior knowledge of geometry, it is worth proving this as a simple exercise.

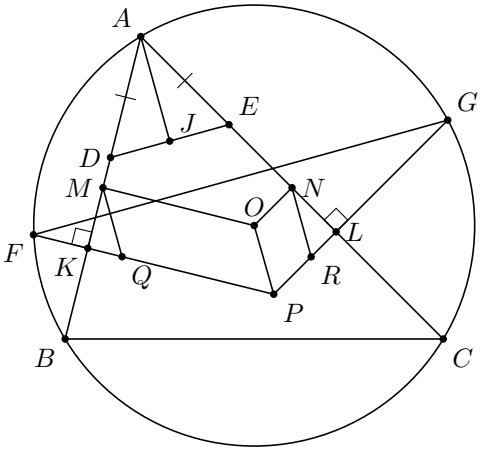


Figure 2: The final configuration.

Triangles MKQ, NLR look congruent.

Claim 2. $MK = NL$

Proof. $MK = BM - BK$; but M, K are defined as midpoints, so $MK = BM - BK = \frac{1}{2}(AB - DB) = \frac{1}{2}AD$.

Similarly, $NL = CN - CL = \frac{1}{2}(AC - EC) = \frac{1}{2}(AE)$.

But $AD = AE$; so $MK = \frac{1}{2}AD = \frac{1}{2}AE = NL$.

Claim 3. $\overrightarrow{MQ} = \overrightarrow{NR}$, and this is parallel to AJ.

Proof. We know $MK = NL$ already; and pretty clearly, $\angle QKM = \angle NLR = 90°$. As MQ, NR are parallel to AJ, $\angle KMQ = \angle DAJ$, and $\angle RNL = \angle JAE$; but $\angle DAJ = \angle JAE$, so $\angle KMQ = \angle RNL$. Thus by SAA congruency, triangles MKQ and NLR are congruent.

As a result, $MQ = NR$. But we know MQ, NR are parallel to AJ, and so MQ is parallel to NR; hence the vector $\overrightarrow{MQ} = \overrightarrow{NR}$, and this is parallel to AJ.

Claim 4. OP is parallel to AJ.

Proof. The vector $\overrightarrow{MQ}$ takes a point on OM to a point on FP; and since these lines are parallel, it must take the line OM to the line FP. Similarly, $\overrightarrow{NR}$ takes a point on ON to a point on GP, so it must take line ON to line GP, as these lines are parallel.

Because $\overrightarrow{MQ} = \overrightarrow{NR}$, this vector takes the intersection of OM, ON to the intersection of FP, GP; thus this vector takes O to P; in other words, $\overrightarrow{MQ} = \overrightarrow{NR} = \overrightarrow{OP}$. But MQ is parallel to AJ; thus OP is also parallel to AJ.

We want to show that FG is parallel to DE; equivalently, we could show FG was perpendicular to AJ. But AJ is parallel to OP; so we can just show FG is perpendicular to OP.

Claim 5. FG is perpendicular to OP.

Proof. We already know that $\angle KMQ = \angle DAJ = \angle JAE = \angle RNL$; but we know OM is perpendicular to KM, and ON is perpendicular to LN. Thus $\angle QMO = 90° - \angle KMQ = 90° - \angle RNL = \angle ONR$.

But $MOPQ, ONRP$ are parallelograms; so $\angle OPQ = \angle QMO = \angle ONR = \angle RPO$. This means that OP is the bisector of $\angle RPQ$; so the line $PQ = FP$ is the reflection of $PR = GP$ in OP.

But since O is the centre of Γ, Γ is its own reflection in OP.

Thus the intersection of Γ and ray PF is the reflection of the intersection of Γ and ray PG in OP; but these points of intersection are F, G respectively. Thus F is the reflection of G in OP.

It follows from this that FG is perpendicular to OP.

Thus FG is perpendicular to OP; but as AJ is parallel to OP, and DE is perpendicular to AJ, DE is also perpendicular to OP; thus FG is parallel to DE.

Problem 2

Find all integers $n \geq 3$ for which there exist real numbers $a_1, a_2, \ldots, a_n + 2$, such that $a_{n+1} = a_1$ and $a_{n+2} = a_2$, and

$$a_i a_{i+1} + 1 = a_{i+2}$$

for $i = 1, 2, \ldots n$.

(Throughout this proof, for all $i > n$, $a_i = a_{i-n}$; this abuse of notation means that for any a_i, we have a_{i-1} preceding it and a_{i+1} following it, and thus the set of a_i has a cyclic structure.)

The first thing one does when presented with this problem is to play around with examples and try to find some series $a_1, \ldots, a_n$ that work. One then discovers, for instance, that if $a_1 = 2, a_2 = 3$, then $a_3 = 7, a_4 = 22, a_5 = 155, \ldots$ and a_i is strictly increasing (and increasing rapidly), and so cannot satisfy $a_{i+1} = a_1$.

One is then motivated to try small values; but while one uses only positive reals, this cannot work[1]. One may try to use 0, but one finds that this always leads to the sequence $0, 1, 1, 2, 3, 7, 22, \ldots$. In fact, one finds that for any positive values one tries, one gets these increasing sequences; and if the sequence $a_i, a_{i+1}, \ldots$ can be shown to keep increasing, then clearly $a_{i+n} > a_{i+n-1} > \cdots > a_{i+1} > a_i$. Then $a_{i+n} > a_i$; but we know $a_{i+n} = a_i$, and we thus have a contradiction.

We therefore formulate the following claim:

Claim 6. There does not exist i such that $a_i, a_{i+1} > 0$.

Proof. Suppose $a_i, a_{i+1} > 0$. Then we find that $a_{i+2} = a_i a_{i+1} + 1 > 1$, and similarly that $a_{i+3} > 1$. Because $a_i, a_{i+1} > 0 \implies a_{i+2} > 1 > 0$, we find, by induction, that $a_j > 1$ for all $j \geq i + 2$.

Suppose $a_j, a_{j+1} > 1$. Then $a_{j+2} = 1 + a_j a_{j+1} > 1 + a_{j+1}$ as a result; we thus find that for all $j \geq i + 2$, $a_{j+1} > a_j$.

1. This is to be proved later.

But then $a_{i+2+n} > a_{i+2+n-1} > \ldots > a_{i+2}$; but we know $a_{i+2} = a_{i+2+n}$, so we have reached a contradiction.

Thus there do not exist $a_i, a_{i+1} > 0$.

We then wonder if 0 might not be of assistance. We quickly realise, however, that if $a_i = 0$, $a_{i+1} = a_{i+2} = 1$; and this is forbidden by the previous claim. So, a_i is never equal to 0.

However, once one notices that negative numbers are also fair game, one identifies, after trying a few examples, that $a_1 = a_2 = -1, a_3 = 2$ provides a solution for $n = 3$; this is also a valid solution whenever n is a multiple of 3[2].

One notices a similar result to that with positive numbers.

Claim 7. There do not exist $a_i, a_{i+1}, a_{i+2} < 0$.

Proof. This is an easy proof. If $a_i, a_{i+1} < 0$, $a_{i+2} = 1 + a_i a_{i+1} > 1 > 0$, so $a_{i+2} \not< 0$.

We have now run out of obvious things to do. The following parts of the proof came to me after several hours and pages of adrenaline-fuelled algebraic scribblings.

Claim 8. There exists i such that $a_i, a_{i+1} < 0$.

Proof. Suppose otherwise. Observe that as there are no two consecutive negative a_i, there must be some positive a_u; without loss of generality, let this be a_1.

As we have assumed that no two consecutive terms are negative, and we have already proven that no two consecutive terms are positive, we know that each term must have different sign from the next one. Thus as a_1 is positive, we find that $a_2, a_4, \ldots$ are negative, while $a_3, a_5, \ldots$ are positive.

(As a result, if n were odd, $n+1$ would be even, so $a_{n+1} = a_1$ would be negative, so we have a contradiction; thus n would be even.)

For all k, $a_{2k-1}, a_{2k+1} > 0$, and $a_{2k} < 0$. Thus $a_{2k+1} = a_{2k}a_{2k-1} + 1 > 0$, and so $a_{2k}a_{2k-1} > -1$, and as we know this to be negative, $|a_{2k}a_{2k-1}| = |a_{2k}||a_{2k-1}| < 1$. We thus find $|a_1||a_2|\ldots|a_n| = (|a_1a_2|)(|a_3a_4|)\ldots(|a_{n-1}a_n|) < 1$.

But also, for all k, $a_{2k+1} > 0$, while $a_{2k}, a_{2k+2} < 0$. Then $a_{2k+2} = a_{2k}a_{2k+1} + 1 < 0$ gives $a_{2k}a_{2k+1} < -1$, so $|a_{2k}a_{2k+1}| > 1$. Then $|a_1||a_2|\ldots|a_n| = (|a_2a_3|)(|a_4a_5|)\ldots(|a_na_1|) > 1$.

Thus we have shown $|a_1||a_2|\ldots|a_n| > 1$ and $|a_1||a_2|\ldots|a_n| < 1$; this is a contradiction. Thus our assumption was wrong, and there exists i such that $a_i, a_{i+1} < 0$.

As I cannot find a way to make the solution I submitted in the IMO seem at all pleasant[3], I shall instead provide the ending of a different proof that I developed later.

2. This is a somewhat worthwhile exercise.
3. It involved making cases on whether $a_i, a_{i+1} < -1$, and ended with several different possible inductions, going in different directions. It nevertheless was a valid solution, gaining full marks, despite being plug-ugly. I am willing to show it to anyone masochistic enough to be interested.

Claim 9. If $a_i, a_{i+1} < 0$, then $a_{i+2} > 0$, and $a_{i+3}, a_{i+4} < 0$.

Proof. Clearly $a_{i+2} = 1 + a_i a_{i+1} > 1 > 0$, and as no two adjacent a_j are positive, $a_{i+3} < 0$.

Observe that as all a_j are not equal to 0, we can divide by them and still have well-defined expressions. Thus as $a_{j+2} - 1 = a_j a_{j+1}$, $a_j = \frac{a_{j+2}-1}{a_{j+1}}$ for all j.

Let us now apply this repeatedly to a_i. We find

$$a_i = \frac{a_{i+2} - 1}{a_{i+1}}$$

$$= \frac{a_{i+2} - 1}{\frac{a_{i+3}-1}{a_{i+2}}} = \frac{a_{i+2}(a_{i+2} - 1)}{a_{i+3} - 1}$$

$$= \frac{\frac{a_{i+4}-1}{a_{i+3}}(\frac{a_{i+4}-1}{a_{i+3}} - 1)}{a_{i+3} - 1} = \frac{(a_{i+4} - 1)(a_{i+4} - 1 - a_{i+3})}{a_{i+3}^2(a_{i+3} - 1)}$$

What sign does this expression have? For a start, as a_{i+3} is negative, $a_{i+3}^2(a_{i+3} - 1)$ is negative, so the expression has the same sign as $-(a_{i+4}-1)(a_{i+4}-1-a_{i+3}) = -(a_{i+4}-1)(a_{i+4}-1+|a_{i+3}|)$.

As $a_{i+4} - 1 = a_{i+3}a_{i+2} < 0$ (and so $-(a_{i+4} - 1)$ is positive), this has the same sign as $a_{i+4} - 1 + |a_{i+3}|$. Observe that as $a_{i+4} > 0$, $-1 < a_{i+4} - 1 < 0$. We need this expression to be negative, as it is equal to a_i, which is negative. We thus need $a_{i+4} - 1 + |a_{i+3}| < 0$, so we need $|a_{i+3}| < 1 - a_{i+4} < 1$. Thus $-1 < a_{i+3} < 0$.

But $-1 < a_{i+4} - 1 < 0$ gives $0 < a_{i+4} < 1$. Thus $-1 < a_{i+3}a_{i+4}$, so $0 < 1 + a_{i+3}a_{i+4} = a_{i+5}$. Thus a_{i+5} and a_{i+4} are both positive; but they are consecutive terms, so this is not possible.

Thus our assumption that $a_{i+4} > 0$ was wrong, so $a_{i+4} < 0$.

Claim 10. n is a multiple of 3.

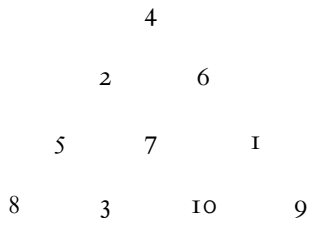
Proof. Select i such that a_i, a_{i+1} are negative; by repeated application of the previous claim, a_{i+3}, a_{i+4} are negative, as are a_{i+3k}, a_{i+3k+1} for all k; and also a_{i+3k+2} is positive for all k.

As a result, f n is not a multiple of 3, then a_{i+2} is positive, but a_{i+n+2} is not. But these are equal; thus n is a multiple of 3.

As $a_1 = a_2 = -1$ gives a solution whenever n is a multiple of 3, and n must be a multiple of 3, these are the only valid values of n.

Problem 3

An anti-Pascal triangle is an equilateral triangular array of numbers such that, except for the numbers in the bottom row, each number is the absolute value of the difference of the two numbers immediately below it. For example, the following array is an anti-Pascal triangle with four rows which contains every integer from 1 to 10.



Does there exist an anti-Pascal triangle with 2018 rows which contains every integer from 1 to $1 + 2 + \cdots + 2018$*?*

One spots fairly easily that the largest number in the triangle must lie on the bottom row, as otherwise there must be two numbers in the triangle which differ by that much, and so one of those two must be larger than the maximum value.

One may consider smaller cases, and observe that any triangle of two rows with 3 on the bottom row is valid.

Both of these, some gazing at the problem, and an understanding of differences suggest the crucial fact that in each small triangle (composed of one number and the two immediately below it), one of the numbers on the bottom row is the sum of the other two.

For each number n in the triangle that is not on the bottom row, let $f(n), g(n)$ denote the numbers directly below it, with $f(n) > g(n)$. Then $f(n) = g(n) + n$, by the definition of the triangle, and so $f(n) > n$.

Where there are r rows in the triangle, and k is the topmost number in the triangle, consider the sequence $k, f(k), f^2(k), \ldots, f^{r-1}(k)$. (Here we use f^x to mean the result of applying f x times; for example, $f^3(k) = f(f(f(k)))$.) This sequence is strictly increasing, as $f(n) > n$ for all valid n, and its terms are underlined in the next figure.

Consider the second sequence $k, g(k), g(f(k)), g(f^2(k)), \ldots, g(f^{r-2}(k))$, the terms of which are circled in the next figure. Observe that the difference between the ith and $i+1$th terms of the first sequence is the $i+1$th term of the second, and consequently, as the first term of both sequences is the same, the ith term of the first sequence is the sum of the first i terms of the second[4].

Here the keen-eyed reader will observe that the numbers circled are those $\leq r$. As the last term of the first sequence is the sum of all the terms of the second, it is minimised when they are the first r numbers; and then it is equal to $1+2+\cdots+r$, its maximum. Thus it cannot be increased beyond this, so the second sequence must be composed of the first r numbers.

So we've found this interesting sequence; what do we do with it? Consider the two sub-triangles with bases being the part of the bottom row to the left, and the part of the bottom row

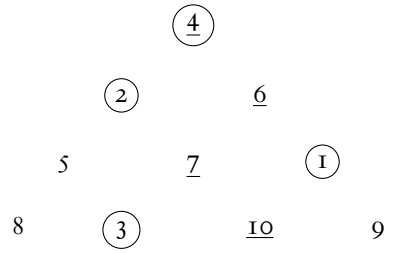


Figure 3: A triangle, with sequences marked.

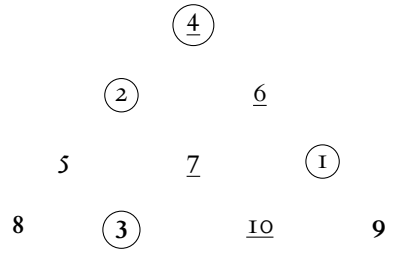


Figure 4: A triangle, with sequences and demilitarised triangles marked in bold.

to the right, of the maximal number in the triangle. Call these the *demilitarised* triangles.

The first sequence does not pass through the demilitarised triangles[5]. As the demilitarised triangles have total base length 2017, at least one has base length $\geq$ 1014. If the second sequence passes through this, it can only pass through the numbers that are adjacent to the rest of the triangle, as all numbers the second sequence are adjacent to numbers in the first sequence. Remove the diagonal column adjacent to the rest of the triangle, and call the remaining triangle the *sanctum*.

The sanctum has side length $\geq 1014 - 1 = 1013$, and contains none of the second sequence. As the second sequence contains all of $1, 2, \ldots, 2018$, all numbers in the sanctum are ≥ 2019.

However, we can define a third and fourth sequence similar to the first and second respectively, but on the sanctum, not the whole triangle. The final term of the third sequence is then the sum of the whole fourth sequence, which must contain 1013 terms, each of which is ≥ 2019.

This final term is minimised when the fourth sequence is $2019, 2020, \ldots$; but then it is $\geq 2018 \times 1013 + \frac{1}{2}(1013 \times 1014)$. But this is greater than $1+2+\cdots+2018$; so such a number is too big for the triangle as a whole. Thus we have a contradiction, and no valid triangle exists.

4. As an exercise, prove this by induction.

5. As an exercise, convince yourself of this, and perhaps prove it.

Phantom pain

Vanessa Ip Yan Lam

Phantom pain is chronic and sporadic pain occurring after the amputation of an organ or limb. A high rate of occurrence of around 80% after limb amputation is estimated.[0] It is also one of the few medical concepts that cannot be fully explained and confirmed by a single mechanism. Our inability to record actual data from the site of pain makes difficult not only the creation of a scale to quantify and compare the pain experienced, but also any justification for treatment, and further investigation.

Since the term 'phantom pain' was coined in the 19th century[1], several attempts have been made to explain this phenomenon. Many physicians first postulated that psychological factors generated before amputation cause such pain to exist after amputation. This includes emotional distress such as fear, depression, and denial.[2] The success of attempts at therapy through progressive relaxation given to amputees demonstrated a degree of correlation between stress level and pain intensity[3]. This is also thought to be valid as in daily life experiences, stress levels indeed affect people's level of pain that can even result in the onset of illnesses such as cramps and migraines[4].

Others suggest a biological explanation. The earliest suggested explanations of phantom pain involved nervous damage and neuroma formation. It is thought that a neuroma must arise from the irregularity of nerves fibres as they attempt to repair themselves post-injury by the proliferation of Schwann cells into a full limb's nerves despite the part itself being physically non-existent. These lesions can cause pain, and so are a potential cause.[5] On a molecular level, such spontaneous activity is also thought to be due to the upregulation of sodium ion channels in the site.[6] Fortunately, as surgical techniques for amputation improve, clinical neuroma formation can be reduced by procedures such as end-to-end nerve coaptation, which involves suturing to an adjacent nerve that is intact.[7] Nevertheless, the incidence of phantom pain remains high amongst amputees. It is therefore questionable whether neuroma formation is a valid explanation of phantom pain. Also, evidence suggested that two-thirds of neuromata formed in the mouth and pharynx do not inflict pain at all.[8] This shows that there is a possibility that neuromata's hypersensitivity might be more of a result of amputation, instead of a definite causative agent of phantom pain itself.

Melzack's argument that 'we do not need a body to feel a body or a physical injury to feel pain' clearly depicts the certainty of perceptions of the phantom limb, and also its pain can be explained biologically. He postulated that a 'neuromatrix' exists in the brain in which the perception of having limbs or organs in amputees is due to the operation of the matrix in the absence of actual external sensory input from the amputated site. This can be due to other signals that enter the matrix, such as the spontaneous action of neurons that 'activated' messages of pain perception which exist due to the ability of the matrix to store 'painful memories'[9].

Melzack's argument was certainly a leap forward, though it remains difficult to determine its validity, due to the complexity of brain wiring. Yet studies based on mapping brain activity and phantom pain perception by magnetic response shows that higher levels of cortical reorganisation result in increased levels of phantom pain. One hypothesis is that such shifts might alter neural circuits that control pain or affect nearby pain centres and result in phantom limb pain. It is also possible that the pain is a result of an imbalance of pain messages due to damage to the nervous system as the amputation is performed.[10]

Another promising approach in the 90s relied upon the existence of 'somatosensory memories' created near the time of amputation. It is thought that phantom pain is largely identical to the sensation experienced by the amputee just before the procedure, as the phantom pain described by the amputees corresponds in detail to their previous pain records[11]. In some cases, amputees even reported having mental images of 'blood trickling down the phantom limb'.[12] At first glance this might seem like a result of psychological memory disorder that causes hallucinations, but it is in fact an example of how the 'vividness of memory components (regardless of "relevance") seems also enhanced by emotional arousal'[13]. The memory complex serves as a trace of information that can lead to the reorganisation

0. Kelly Patrick Anthony Byrne. "Survey of Phantom Limb Pain, Phantom Sensation and Stump Pain in Cambodian and New Zealand Amputees". In: *Pain Med* 12.5 (May 1, 2011), pp. 794–798. ISSN: 1526-2375. DOI: 10.1111/j.1526-4637.2011.01105.x. URL: https://academic.oup.com/painmedicine/article/12/5/794/1910422 (visited on 07/19/2018).
1. Silas Weir Mitchell. *Injuries of Nerves and Their Consequences*. J.B. Lippincott, 1872. 396 pp.
2. Joel Katz and Ronald Melzack. "Pain 'Memories' in Phantom Limbs: Review and Clinical Observations". In: *Pain* 43.3 (Dec. 1, 1990), pp. 319–336. ISSN: 0304-3959. DOI: 10.1016/0304-3959(90)90029-D. URL: http://www.sciencedirect.com/science/article/pii/030439599090029D (visited on 07/19/2018).
3. John G. Arena et al. "The Relationship between Situational Stress and Phantom Limb Pain: Cross-Lagged Correlational Data from Six Month Pain Logs". In: *Journal of Psychosomatic Research* 34.1 (Jan. 1, 1990), pp. 71–77. ISSN: 0022-3999, 1879-1360. DOI: 10.1016/0022-3999(90)90009-S. pmid: 2313614. URL: https://www.jpsychores.com/article/0022-3999(90)90009-S/abstract (visited on 07/19/2018).
4. Khara M. Sauro and Werner J. Becker. "The Stress and Migraine Interaction". In: *Headache: The Journal of Head and Face Pain* 49.9 (2009), pp. 1378–1386. ISSN: 1526-4610. DOI: 10.1111/j.1526-4610.2009.01486.x. URL: https://onlinelibrary.wiley.com/doi/abs/10.1111/j.1526-4610.2009.01486.x (visited on 07/19/2018).
5. Julei C. Fanburg-Smith. "Nerve Sheath and Neuroectodermal Tumors". In: *Bone and Soft Tissue Pathology*. Ed. by Andrew L. Folpe and Carrie Y. Inwards. Elsevier Health Sciences, Jan. 1, 2010, pp. 193–238. ISBN: 978-0-443-06688-7.
6. L. Nikolajsen and T.S. Jensen. "Phantom Limb Pain". In: *British Journal of Anaesthesia* 87.1 (July 2001), pp. 107–116. ISSN: 00070912. DOI: 10.1093/bja/87.1.107. URL: https://linkinghub.elsevier.com/retrieve/pii/S0007091217363481 (visited on 07/19/2018).
7. Mohammad M. Al-Qattan. "Prevention and Treatment of Painful Neuromas of the Superficial Radial Nerve by the End-to-Side Nerve Repair Concept: An Experimental Study and Preliminary Clinical Experience". In: *Microsurgery* 20.3 (2000), pp. 99–104. ISSN: 1098-2752. DOI: 10.1002/(SICI)1098-2752(2000)20:3<99::AID-MICR1>3.0.CO;2-Y. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/%28SICI%291098-2752%282000%2920%3A3%3C99%3A%3AAID-MICR1%3E3.0.CO%3B2-Y (visited on 07/19/2018).
8. Douglas R. Gnepp. *Diagnostic Surgical Pathology of the Head and Neck E-Book*. Elsevier Health Sciences, Apr. 7, 2009. 1225 pp. ISBN: 978-1-4377-1951-2.
9. Ronald Melzack. "Phantom Limbs". In: *Scientific American* 266.4 (1992), pp. 120–127. ISSN: 0036-8733. JSTOR: 24939024.
10. B. Bower. "Brain changes linked to phantom-limb pain." In: *Science News* 147.23 (1995), p. 357. ISSN: 0036-8423.
11. Katz and Melzack, op. cit.
12. W. R. Henderson and G. E. Smyth. "PHANTOM LIMBS". in: *J Neurol Neurosurg Psychiatry* 11.2 (May 1948), pp. 88–112. ISSN: 0022-3050. pmid: 18861109. URL: https://www.ncbi.nlm.nih.gov/pmc/articles/PMC497107/ (visited on 07/19/2018).
13. German E. Berrios and Mauricio Sierra. "Flashbulb and Flashback Memories". In: *Memory Disorders in Psychiatric Practice*. Ed. by German E. Berrios and John R. Hodges. Cambridge University Press, Mar. 16, 2000. ISBN: 978-0-521-57671-0.

of the S1 cortex. This can be due to nerve sprouting or the activation of silent connections that can lead to pain as the spontaneous activity of the nerve that supplies the former region of the limb or organ arises.[14] Under this hypothesis, the mechanism of phantom pain is not merely pscyhological. However, as there are clear missing links between the memory complex's formation and the biological alterations, it is still open to debate. In fact, studies on distraction therapy point more to a psychological explanation, as the amputee claimed that she can 'switch off' her phantom pain by distracting herself after training[15]. This parallelsother studies on the effectiveness of cognitive behavioural therapy in chronic pain patients which reported improvements in pain and disability.[16] Under this scenario, it seems that the psychological aspects of the mechanisms causing phantom pain have a biological basis. This demonstrates how the two seemingly distinct mechanisms collectively constitute a single valid mechanism..

But in order to verify the contribution of biological factors to changes in the nervous system, greater detail and more experimental data are needed. This is particularly important as there are already many existing hypotheses on how phantom pain arises, but many lack a detailedexplanation of their proposed idea. Yanagisawa et al.[17] investigated the possible relationship between phantom pain and the reorganisation of the sensorimotor cortex by training a brain machine interface (BMI) to decode neural activity of the brain, thereby revealing associated movement of the amputated site. Then the signal is converted to a robotic neuroprosthetic BMI in real time. This can induce plastic change in the cortex that can affect the levels of phantom pain. Results show that associating the prosthetic hand with the neural activity related to the phantom hand increased pain, whilst associating it with the intact hand reduced pain and also decreased the quantity of between phantom pain and sensorimotor cortical plasticity. In fact, this is a rather surprising discovery as many believe that learning to associate the phantom limb with the prosthetics should alleviate pain, since functional restoration would occur and reorganisation through such means would help the body 'feel' as if the amputated site is really present with functionality. The rise in pain demonstrates phantom pain. This discovery seems to be related to the effectiveness of 'mirror therapy' as the amputees are trained to associate their intact limb movement with the phantom limb using a mirror, so the resulting cortex reorganisation is likely related to the intact hand which reduced pain. Indeed, in an investigation on mirror therapy, 100% of the participants in the mirrored therapy group reported a decrease in phantom pain, whilst 67% of the participants in the mental visualization group reported an increase in pain.[18] This demonstrates a clear distinction between mental visualization and mirror therapy as mental visualization relies on changes in the reorganisation of the cortex related to the amputated site, whilst mirror therapy relies on the intact limb as a mirror image that potentially causes mirror neurons to reorganise. This result supports previous findings through imaging, which established that it would be worth investigating cortex reorganisation and plasticity..

As mentioned previously, another way of thinking about phantom pain is to avoid treating biological and psychological explanations as dichotomous. This is because even psychological factors can also be mediated by biological means. After major traumatic events or abusive events over a long period, post-traumatic stress disorder can develop due to the dysregulation of the CNS.[19] The patient will constantly be in a state of intense fear and helplessness and nociceptor excitation can occur. This can lead to incidence of onset fibromyalgia[20] and other chronic pain. Related to phantom sensations, this might be the missing link between phantom pain and traumatic experiences that lead to amputation. But since not all amputations are carried out due to traumatic experiences such as accidents and medical conditions, this hypothesis cannot be generalised as a definite cause of phantom pain.

One question that should be raised is whether emotional and psychological are a cause or consequence of such pain. This is because most analyses on phantom pain being a psychological issue build on the trauma and distress of amputees. Therefore, along with the pain, psychological problems are also seen as an effect post-amputation. This observation may cause people to intuit causation. It is difficult to rule out the possibility that such psychological problems are a consequence of phantom pain, rather than the cause. A study of chronic pain also examined such relationship and reached the conclusion of emotional disturbance as 'more likely to be a consequence than a cause of chronic pain'.[21] This strengthens the argument that psychological problems are unlikely to be the cause of phantom pain, despite differences between.

Stumps are a key site of amputation which could be closely related to the cause of phantom pain. Some have investigated its relationship with phantom pain itself, as thesweating and vasoconstriction at the stump are involved in phantom pain as they can 'spread' to the phantom limb if perceived to

14. H. Flor. "Painful Memories: Can We Train Chronic Pain Patients to 'forget' Their Pain?" In: *EMBO Reports* 3.4 (Apr. 15, 2002), pp. 288–291. ISSN: 1469221X, 14693178. DOI: 10.1093/embo-reports/kvf080. URL: http://embor.embopress.org/cgi/doi/10.1093/embo-reports/kvf080 (visited on 07/19/2018).
15. Marteinn Steinar Jonsson and Keren Fisher. "Phantom Pain—Psychological Conceptualization and Treatment: A Case Report". In: *Behavioural and Cognitive Psychotherapy* 24.3 (July 1996), pp. 275–281. ISSN: 1469-1833, 1352-4658. DOI: 10.1017/S1352465800015125. URL: https://www.cambridge.org/core/journals/behavioural-and-cognitive-psychotherapy/article/phantom-painpsychological-conceptualization-and-treatment-a-case-report/14DEC377B2F301CE213D1AEEECA58FD3 (visited on 07/19/2018).
16. Judith A. Turner. "Comparison of Group Progressive-Relaxation Training and Cognitive-Behavioral Group Therapy for Chronic Low Back Pain". In: *Journal of Consulting and Clinical Psychology* 50.5 (1982), pp. 757–765. ISSN: 1939-2117(Electronic),0022-006X(Print). DOI: 10.1037/0022-006X.50.5.757.
17. Takufumi Yanagisawa et al. "Induced Sensorimotor Brain Plasticity Controls Pain in Phantom Limb Patients". In: *Nature Communications* 7 (Oct. 27, 2016), p. 13209. ISSN: 2041-1723. DOI: 10.1038/ncomms13209. URL: https://www.nature.com/articles/ncomms13209 (visited on 07/19/2018).
18. Brenda L. Chan et al. "Mirror Therapy for Phantom Limb Pain". In: *New England Journal of Medicine* 357.21 (Nov. 22, 2007), pp. 2206–2207. ISSN: 0028-4793. DOI: 10.1056/NEJMc071927. pmid: 18032777. URL: https://doi.org/10.1056/NEJMc071927 (visited on 07/19/2018).
19. Samantha Meltzer-Brody et al. "Trauma and Posttraumatic Stress Disorder in Women With Chronic Pelvic Pain". In: *Obstetrics & Gynecology* 109.4 (Apr. 2007), p. 902. ISSN: 0029-7844. DOI: 10.1097/01.AOG.0000258296.35538.88. URL: https://journals.lww.com/greenjournal/Fulltext/2007/04000/Trauma_and_Posttraumatic_Stress_Disorder_in_Women.17.aspx (visited on 07/19/2018).
20. Manfred Zimmermann. "Pathophysiological Mechanisms of Fibromyalgia". In: *The Clinical Journal of Pain* 7.1 (Mar. 1991), S8. ISSN: 0749-8047. URL: https://journals.lww.com/clinicalpain/abstract/1991/03001/pathophysiological_mechanisms_of_fibromyalgia.3.aspx (visited on 07/19/2018).
21. Ann Gamsa. "Is Emotional Disturbance a Precipitator or a Consequence of Chronic Pain?" In: *Pain* 42.2 (Aug. 1, 1990), pp. 183–195. ISSN: 0304-3959. DOI: 10.1016/0304-3959(90)91161-B. URL: http://www.sciencedirect.com/science/article/pii/030439599091161B (visited on 07/19/2018).

exist..[22] The mechanism behind, however, is relevant to most neurological arguments made on the existence of phantom pain. It is hypothesised that such burning pain is evidence that the sympathetic nervous system is the cause, and hyperactivity of its regional pathways due to vascular alterations could be a cause of such involvement.[23]

Nevertheless, it is fundamentally important to distinguish between pain due to amputation and phantom pain, as the resultant pain can be due to stump pain in the amputated area instead of pain on the amputated, non-existent area. Therefore, the fact that amputees feel that their phantom limb is really present physically or even visually reveals an element of psychological involvement.

Proponents of both psychological and biological explanations have attempted to theorise the mechanism of phantom pain as a single action. However, treating it as such a straightforward neurological disorder might be to ignore the unique nature of phantom pain as being present in a non-present part of a body. With this in mind, I believe that phantom pain arises from multiple mechanisms that are both neuronal and psychological. These factors can also affect the intensity and duration of phantom pain at later stages, so potential therapies should be investigated on this basis. This includes the fact that psychological problems can be a consequence of phantom pain, which intensifies the pain itself.

Phantom pain is a thought-provoking phenomenon that has reshaped assumptions made based on observations and experimentation. Drawing clear lines via the concept of 'mind-body dualism' between psychological factors and biological factors seems to be a general approach to maintain the 'rigidity' of biological pathways. Ultimately, psychological mechanisms do not solely operate in perception, but rather determine how stimuli are processed, just like other biological pathways. Despite the difficulty in weighing the importance of both factors comparatively, one can establish the input of both psychological and biological bases in the existence of phantom pain. Research on connexions between the two could prove to be fascinating. There could, for example, be circular mechanisms, such as psychological stress that increases steroid hormone levels that can influence neurotransmission in the CNS and PNS.[24] This might be a source of pain perception in the amputated site and should be further investigated. In conclusion, the multiple mechanisms of phantom pain can be viewed in a discrete and scientific manner, whilst they can be treated as a cross-linked system. With efforts to consider the interactions between the two, such analysis could certainly help to better tackle phantom pain.

22. W. K. Livingston. "FANTOM LIMB PAIN: A REPORT OF TEN CASES IN WHICH IT WAS TREATED BY INJECTIONS OF PROCAINE HYDROCHLORIDE NEAR THE THORACIC SYMPATHETIC GANGLIONS". in: *Arch Surg* 37.3 (Sept. 1, 1938), pp. 353–370. ISSN: 0272-5533. DOI: 10.1001/archsurg.1938.01200030002001. URL: https://jamanetwork.com/journals/jamasurgery/fullarticle/543686 (visited on 07/19/2018).
23. Joel Katz. "Psychophysiological Contributions to Phantom Limbs , Psychophysiological Contributions to Phantom Limbs". In: *Can J Psychiatry* 37.5 (June 1, 1992), pp. 282–298. ISSN: 0706-7437. DOI: 10.1177/070674379203700502. URL: https://doi.org/10.1177/070674379203700502 (visited on 07/19/2018).
24. Katy Vincent and Irene Tracey. "Hormones and Their Interaction with the Pain Experience , Hormones and Their Interaction with the Pain Experience". In: *Reviews in Pain* 2.2 (Dec. 1, 2008), pp. 20–24. ISSN: 2042-1249. DOI: 10.1177/204946370800200206. URL: https://doi.org/10.1177/204946370800200206 (visited on 07/19/2018).

When and why did Christians and Rabbinic Jews stop considering themselves to belong to the same religion?

Lara Brown

It was inevitable from the moment Jesus began his ministry that there would be a textual conflict between Christianity and Rabbinic Judaism. In particular, there was the overwhelming issue of Jesus' messianic status. As neither a warrior nor someone who brought peace or salvation to the Jewish people, it was impossible to even consider his claim to be the messiah. The very manner of Jesus' death—on the cross, as a common criminal—presented another stumbling block. Furthermore, Jesus' attitude to the Old Testament had the potential to be hugely problematic; he argued himself to be fulfilling and through some interpreters eyes, superseding the laws of the Torah. Mathew 5:17 highlights this tension when Christ argued: '[d]o not think I have come to abolish the Laws or the Prophets; I have not come to abolish them but to fulfil them'[0]. This caused followers of Christ to view the laws of Moses through a completely different lens. These theological differences could, at first interrogation, appear to make Rabbinic Judaism and Christianity irreconcilable from the moment Christianity was established. However, the two sects appeared to coëxist throughout, at the minimum, the second temple period. At this early point in Christian history the Jews were described as merely baffled by Christians' belief in Jesus' messiahship. Rabbinic literature barely addressed Christianity, presenting it as an irrelevant sect.

Due to the coëxistence of the two religions in the first century despite these theological contradictions, this essay his given less weight to a view of textual conflict as the source of the split, instead considering the two religions' perceptions of each other, and when and how they dealt with and recognised underlying textual differences, not when and how these differences arose. In particular, recognition of these differences seems not to have occurred during the first and possibly the second century, as there was an seemingly blurred division between the two sects.

In answering this question, the period which will be examined ranges from 64 C.E. when the Christians were persecuted by Nero as a distinct group after the great fire of Rome, to 170 C.E. as Josephus ends his narrative. It may appear unusual that these chosen dates extend so far into the second century, as it is often assumed that the division was complete by the end of the second temple period, with the sack of Jerusalem in 70 C.E. acting as a consolidating point. This fails to recognise the blurred lines between the two groups which existed long after the Jewish-Roman war and overestimates the significance of the fall of the temple.

Judaism in the period addressed by this essay is too diverse to be dealt with as a distinct entity. Rabbinic Judaism is therefore the focus. Throughout the first and second centuries it was a way of life formulated by the rabbis with the majority of its written sources finalised after around 100 C.E. onwards. Its

0. Matt. 5:17 (Kenneth L Barker, ed. *NIV Study Bible: New International Version*. OCLC: 945214539. Grand Rapids, Mich.: Zondervan, 2008. ISBN: 978-0-310-93916-0)

teachings were rooted in the Hebrew Bible and it is widely considered the foundation of all forms of Judaism which exist today. Rabbinic Judaism typically dealt with the Torah as well as the 'oral law' (tradition). When dealing with rabbinic sources, several contentious issues arise. They tend to address a very limited range of issues and very rarely regard or acknowledge contemporary historical events. Neither the Jewish-Roman wars nor the rise of Christianity are extensively engaged with. This is likely because they were written chiefly for internal consumption. Rabbinic texts are also likely to have been censored by both Jews and Christians, meaning the few original allusions to Jesus and the Christians in the texts may have been removed by the Jews for their own protection. It is further important to remember that Rabbinic Judaism was not representative of Judaism at the time; it has been chosen for this study because after the Jewish war it became the dominant and representative strain. It is also arguably the only group possible to study, as they authored the few sources we have.

The traditional view of the Judæo-Christian split views the Jewish-Roman war as the primary cause of the divide. A. von Harnack argues that '[i]t was the destruction of Jerusalem and the temple which seems to have provoked the final crisis, and led to a complete breach between the two parties.'[1] The Roman persecution of the Jews which followed the temple's fall is often seen to have motivated Christians to distance themselves from the Jewish faith so as to protect themselves. Further, the two strains of belief interpreted the cause of the temple's destruction differently. Christians saw the fall of the temple to be a rejection of the Jews by a vengeful God and a punishment for the killing of Christ; Jews, on the other hand, viewed it as a correction by a still loving God for their failure to adequately follow his laws. Moreover, the sack of Jerusalem is believed by many scholars to have marked the beginning of a period of oppression of the Jews by the Romans, leaving adherence to the Jewish sect by Christians less appealing. This scholarly tradition is reliant on the view that the sack of Jerusalem was calamitous, which is certainly true to an extent. We know that thousands were held captive, sold into slavery and killed, large parts of the city and of course the temple (a place many Jews felt God supremely dwelt) were destroyed, Judea ceased to be any sort of political entity, and the Roman hold over it was tightened. Josephus's *Bellum Judaicum* tells us that '[Caesar] did assign to eight hundred veterans discharged from the army a place of habitation called Emmamaus, distant thirty furlongs from Jerusalem'[2]. Furthermore, the reach of the temple tax was extended. These impacts are recorded in Ezra 4, Baruch 2, and Josephus's Bellum Judaicum.[3]

Academics such as G. Alon[4] argue the war was less impactful: he notes that the rest of Judea had already been pacified leaving the final siege to only affect Jerusalem, that there was no change in the legal status of the Jews and that the diaspora was barely affected. Jewish observance, Alon remarks, was not banned and neither was the rebuilding of the temple. Alon argues that the Jews suffered at the hands of the Romans briefly, but were able to restore their relationship fairly rapidly and without lasting damage. This left them in a comparatively strong political position, with the umbrella of Jewish privilege and protection remaining an appealing one to Christians. The Fiscus Judiacus, a tax introduced by Vespasian after the sack of Jerusalem in 70 c.e. is often pointed to as further motivation for the Christians to exercise self-determination in order to demonstrate allegiance to Rome and avoid the tax. However, it is important to note that even at the height of the tax and other Roman oppression of Judaism, the status of the Jews was still far higher than that of the Christians. Profession of Christianity at this time was outlawed and Christians were routinely exiled, as Eusebius tell us: '[i]n the fifteenth year of Domitian, Flavia Domitilla … was banished with many others to the island of Pontia as testimony to Christ'[5]. In the immediate wake of the Jewish-Roman war Judaism still offered Christians a wealth of privileges and protections which compensated for the tax. As G.W.H. Lampe observes, '[t]he capture of Jerusalem by Titus and the burning of the Temple seem, so far as we can judge from the literature of the succeeding century and a half, to have made a surprisingly small impact on the Christian communities'[6].

An often overlooked consequence of the Jewish-Roman war is that the Yavnean sages were able to found their academy with Roman approval. A popular line of argument follows that after 70 c.e. the actions of the rabbis at Yavneh showed increasing hostility towards the Christians, manifesting in the prohibition of Christian books, closure of the cannon, banishment from the synagogue, as observed in John 8:22 'the Jewish leaders, who already had decided that anyone who acknowledged Jesus was the Messiah would be put out of the synagogue'[7] and John 12:42 'because of the Pharisees they would not properly acknowledge their faith for fear they would be put out of the synagogue'[8], and cursing in the liturgy. Further, the development of the *sifre minim* ('heretical books'), which were said to cause contamination and had to be burnt has been seen as evidence for the specific targeting of Christian modes of worship. Heretics were considered worse that Pagan idolaters because they knew God but denied him. In the past the *sifre minim* have been taken to have been exclusively Christian books, although further examination of rabbinic usage has shown that *minim* is a general word for heretic and would not have been limited to Christians. Gnostics, apocalyptics and Helenizers would have also been targeted by the *sifre minum.* Here, it becomes important to recognise that the *sifre minum* was not about distancing the Jewish faith from Christianity, but simply preserving its purity. The measures taken by the Yavnean Sages were necessitated by the damage to Jewish identity caused by the war with Rome. Stephen G. Wilson argues that the negative impact on Christians as well as the degree to which they were impacted by these doctrines has likely been overstated: '[i]n recent works, however, there has emerged a

1. Adolf von Harnack and James Moffatt. In: *The Mission and Expansion of Christianity in the First Three Centuries*. OCLC: 760528879. New York: Harper, 1962, p. 63.
2. Flavius Josephus, Paul L Maier, and Flavius Josephus. "Bellum Judiaicum 7.216". In: *Josephus, the Essential Works: A Condensation of Jewish Antiquities and the Jewish War*. OCLC: 30544429. Grand Rapids, Mich.: Kregel Publications, 1994. ISBN: 978-0-8254-3260-6.
3. Michael E. Stone. "REACTIONS TO DESTRUCTIONS OF THE SECOND TEMPLE Theology, Perception and Conversion". In: *Journal for the Study of Judaism* 12.2 (Jan. 1, 1981), pp. 195–204. ISSN: 0047-2212, 1570-0631. DOI: 10.1163/157006381X00126. URL: http://booksandjournals.brillonline.com/content/journals/10.1163/157006381x00126 (visited on 08/29/2018).
4. Gedaliah Alon and Gershon Levi. In: *The Jews in Their Land in the Talmudic Age (70-640 C.E.)*. OCLC: 248540331. Cambridge, Mass.: Harvard Univ. Press, 1996, pp. 1–17. ISBN: 978-0-674-47495-6.
5. Eusebius and Christian Frederic Crusé. *Eusebius' ecclesiastical history: complete and unabridged*. OCLC: 234081458; 3.18.4. Peabody, Mass.: Hendrickson Publishers, 2006. ISBN: 978-1-56563-813-6.
6. G.W.H. Lampe. "A.D. 70 in Christian Reflection". In: *Jesus and the Politics of His Day*. Ed. by E. Bammel and C. F. D. Moule. Cambridge University Press, Aug. 30, 1985, pp. 153–171. ISBN: 978-0-521-31344-5.
7. John 8:22 (Barker, op. cit.)
8. John. 12:42 (ibid.)

largely persuasive consensus that the anti-Christian motives of the Yavnean rabbis have been considerably exaggerated'[9].

The banning of texts was known as the *Birkat ha-minim* ('benediction against heretics'). The story of the malediction's origin dates back to around 80-95 C.E. and was recorded in rabbinic writings: 'Simon ha-Pakuli ordered the Eighteen Benedictions before Rabban Gamaliel in Yavneh. Rabban Gamaliel said to the sages: Is there no one who knows how to compose a benediction against the *minim*? Samuel Ha-Qatan stood up and composed it'[10]. These, paired with Johannine references to the Christians being cast out from the synagogue, would support the idea that a major schism took place around this point; just after the sack of Jerusalem. However, it is important not to overlook the fact that the only Christians criticised by Yavnean sages would have been Jewish; people who were increasingly in the minority. Further Stephen G. Wilson emphasises that '[t]hese teachings were designed not to attack the Christians but to preserve the stability and purity of the Jewish community'[11].

The apparent rejection of much of the fabric of Christianity as *sifre minum* may lead some to assume that a definitive split had taken place by around 85 C.E. Pliny's correspondence with Trajan, however, highlights the lack of clarity of any such division or rejection. Pliny wrote to Trajan in around 112 C.E. after a number of Christians were brought before him on charges lodged (often anonymously through pamphlets) by the local populace. He mentions that he was not present at the trial of the Christians, informing the reader that one occurred. He further tells Trajan: 'I do not know the methods or the limits to be observed either in examining or punishing them [Christians]'[12]. Pliny outlines his method for the identification of Christians, explaining: 'I asked them personally whether they are Christians, if they confessed it, I repeated the question a second and a third time, adding the threat of capital punishment; those who persisted, I ordered to be executed.'[13] The crudeness of this method highlights the difficulties of identifying Christians and demonstrates that the Romans were far more preoccupied with punishing defiance of the party line than actual Christian belief. In his letter to Trajan, Pliny identifies three potential types of Christian: those who confess to Christianity; those who deny it; and those who confess to having been Christians but have since defected. Pliny is certain that the first should be executed (unless they are Roman citizens—a further sign of the uncertain identity of Christianity) and the second set free. He is writing to Trajan for advice on his attitude to the third. Trajan's response is that Christians should be executed unless they recant, leaving the third group Pliny identified (those who have defected) safe. Trajan also stressed the Pliny should ignore anonymous charges found in pamphlets and that 'no search should be made for these people'[14]. The Romans see nothing particularly unique about Christians and are not interested in seeking them out so much as punishing defiance when they are confronted with it. It is also important to note that Christianity, unlike Judaism, was outlawed, leaving those of the Christian faith with motivation to attach themselves to Judaism on account of the protection on it afforded them, delaying the split further.

One major risk, in the writing of this essay, is the tendency to answer the question from a Christian position. This danger emerges from the abundance of Christian sources in comparison to rabbinic ones. One could therefore easily consider only the changes in Christianity and the theological disputes which emerge from this. It is crucial, however, to consider the evolution of Rabbinic Judaism after the establishment of the Javneh Sanhendrin.

When one considers the state of Judaism in the first century, the traditional rabbinic view of Christianity as a paganized form of Jewish ethical monotheism starts to fall apart. In order to gain a better understanding of the position of first century Judaism it is helpful to analyse the genesis of Christianity; crucially, one must recognise that Jesus' ministry was just one of many influences—both Judaism and Paganism were arguably equally important. The Judaism which acted as a key source of early Christianity was not the consolidated and clearly monotheistic religion which was finalised and secured at the Yavneh Sanhendrin in the second century. Israelite polytheism was far more pervasive; this championed the view that 'God was one, eventually the king, of a larger pantheon, a king who had reigned with a divine queen, who had a son, or been a son'[15]. This polytheistic view is likely to be the source of Christianity's trinity. As Segal notes, '[h]idden within the texts [rabbinic writings] is the Jewish witness to the rise of Christianity…they indicate that the nascent Christian faith began to differentiate and define itself somewhere on the evolving continuum from earlier pluralistic Judaism to radical Gnosticism'[16]. It is only when Rabbinic Judaism moved away from this earlier pluralistic Judaism, rejecting it in the Orthodox synthesis at Yavneh Sanhendrin, that Christianity began to look separate. By observing changes to Judaism and lifting the focus of exclusively Christianity's evolution, one can locate the split between the two religions much later, in the second century.

One of the most obvious signs that actual division had taken place appeared in Christian apologetic writing. Here, it is important to remember that Christianity was outlawed. The majority of these writings were therefore written to remain within the community; in the cases where they were read by Roman officials their authors tend to have been executed. Three apologetic writings which depict a real shift in the Christian attitude towards the Jews are the *Epistle to Diognetus*, *The First Apology* and *Dialogue with Trypho.* These were all written in the late second century and demonstrate an attitude of aggressive hostility towards the Jewish faith. *Epistle to Diognetus* tends to be dated between around 130-200 C.E. and immediately poses the question of why Christians do not worship in the same manner as Jews. The author dismisses the Jewish people as foolish for offering sacrifices to a God who does not need them and argues them to be impious for favouring some days, months and foods over others when God created all things equal. *The First Apology,* generally dated between 150-155 C.E., is even more hostile, positing that 'Jews are the enemy, denying Jesus and persecuting Christians' and

9. Stephen G Wilson. *Related Strangers: Jews and Christians, 70-170 C.E.*. OCLC: 148683933. Fortress Pr, 2006. ISBN: 978-0-8006-3733-0, p. 176.
10. T. Berakhot 28b-29a (H Polano. *The Talmud.* OCLC: 9391622. London; New York: F. Warne, 1965)
11. Wilson, op. cit., p. 178.
12. Ralph Martin Novak. *Christianity and the Roman Empire: Background Texts.* Harrisburg, Pa: Trinity Press International, 2001. 340 pp. ISBN: 978-1-56338-347-2, p. 47.
13. Ibid., p. 47.
14. Ibid., p. 49.
15. Margaret Barker. *The Great Angel: A Study of Israel's Second God.* OCLC: 694851000. Louisville: Westminister John Knox Press, 1992. ISBN: 978-0-664-25395-0.
16. Wilson, op. cit., p. 191.

that 'despite their devastating defeats by the Romans they do not understand their own fate'[17]. *Dialogue with Trypho* dated between 155–156 c.e. expresses the same sentiment in the form of a Socratic debate. There is a clear change in the Christian attitude towards Rabbinic Judaism towards the end of the second century. Decisive action to alienate themselves from Judaism appears to be taken by Christians. This sudden shift in attitude would suggest the point of the split's finality to be the first half on the second century. The question now arises: which events provoked this final separation?

The Bar Cochba rebellion in 133–135 c.e. is one good candidate. The revolt was by the Jews against Hadrian, probably in reaction to a ban on circumcision and the building of Aelia Capitolina, a Pagan colony on the site of Jerusalem. Their goal was to overthrow the Romans and reëstablish an independent Judea. The potential restoration of the Temple was another key motive of the revolt, highlighting the importance of the Sack of Jerusalem in the narrative of the split. This goal was epitomised by the Bar Cochba coins which spoke of the liberation of Jerusalem. Here it is important to note that Hadrian would have been unlikely to have been trying to aggravate or persecute the Jews; the wealth of concessions made to the Jewish people by the Roman Empire at this time demonstrates this. The ban on circumcision was part of a wider ban on castration which the Romans viewed as barbaric, and the building of Aelia Capitolina was not intended as an affront to Jerusalem. During the initial rebellion Bar Cochba came to be viewed as a messianic figure; born Simon ben Kosevah his name was changed in the literature to Bar Cochba, 'son of star'. After the revolt failed Rabbinic writing often refers to him as Bar Coziba, 'son of lies'[18]. There is ample evidence to suggest Bar Cochba attempted to force Christians to accept his messianic status, requiring them to deny Jesus. Those who refused to do this were executed. This is the first evidence of Christian belief in Jesus' messiahship causing serious friction with Judaism. *The Apocalypse of Peter* records that '[t]hey will promise that "I am Christ who has come into the world"…But this liar was not Christ and when they have rejected him he will kill with his sword and many will become martyrs'[19]. Bar Cochba's claims forcibly brought to light this key point of dispute between Jews and Christians for the first time. In 132 c.e. Bar Cochba established an independent Jewish state. He ruled as Nasi—'Prince'—for three years before its conquest by the Romans in 135 c.e. . After the failed rebellion the Jews faced a period of oppression far heavier than anything they had experienced before. They were banned from Jerusalem and there is evidence in Rabbinic literature that Jewish observance was banned, including observance of the Sabbath[20], the ordination of rabbis[21] and the study of the Torah[22]. The date the schism, which had been in progress since the sack of Jerusalem, was finalised, is arguably the point at which this oppression began in 135 c.e. In refusing to recognise Bar Cochba as the Messiah the Christians were directly persecuted by the Jews for the first time. Further, the oppression of the Jews by the Romans left them unattractive as a group for the Christians to ally themselves with. As Lampe argues, the events of the Jewish war were 'remembered in association with, and to some extent only as a prelude to, the ever more final and crushing judgement of God executed in 135 against the opponents of the church's claim to be the authentic Israel'[23].

The addition of the *notzrim*[24] (Nazerenes, i.e, Christians) to the malediction in what many scholars have suggested was around 135 c.e. significantly strengthens the case for the Bar Cochba revolt. At this point the Christian church would have been increasingly gentile meaning they would no longer have fallen under the *minum* (curse against heretics). Urbach suggests that Nazarenes would have needed to have been added to those included in the *minum* after the Bar Cochba revolt as the Jewish leadership of the Jerusalem church would have been replaced by gentiles, demonstrating the increasingly gentile nature of the Christian church and exemption of the vast majority of Christian liturgy from the old malediction. This was symbolic of a major turning point in Judæo-Christian relations, caused by the Bar Cochba revolt and subsequent oppression. It is the first example of a formal, liturgical malediction, approved by Yavnean leaders and read out in synagogues to specifically condemn Christians.

It is easy to adopt the simple narrative that the division between Rabbinic Judaism and Christianity was finalised by the fall of the temple, addressing the sack of Jerusalem as a final and decisive event. This approach must be avoided due to its lack of appreciation for the nuanced relationship between Judaism and Christianity throughout much of the second century. There is an undeniable shift in Judæo-Christian relations after the fall of the Temple, but this change is not sufficiently definitive to be labelled as a split. There are some instances of Christians being identified as a distinct group before 70 c.e., Nero's persecution of the Christians by way of scapegoat after the Great Fire of Rome in 64 c.e. being the most prominent example. This provides evidence that Christians could be identified by both the local people they were disliked by and Nero's régime at a very early point. It does not demonstrate that Christians were in any way separate from Judaism. The ongoing changing relationship between Christianity and Judaism after 70 c.e. is evidenced by the increasingly hostile attitude adopted by the Yavnean Sages, who congregate as a consequence of the sack of Jerusalem. Despite this, the temptation to view the division as complete after or even solely caused by the Jewish-Roman war must be avoided. There was no clear or decisive separation at that early point and Pliny's letter clearly demonstrates that Christians were yet to establish a unique identity. The point when the split became irreversible is therefore most likely to have been after the Bar Cochba revolts. This was the first time Jews actively persecuted Christians, and the addition of the *notzrim* malediction marks the first formal liturgical separation from Christianity by Rabbinic sources. Furthermore, Christian apologetic writings after 135 c.e. support this view of separation.

17. Iustinus. *St. Justin Martyr: The First and Second Apologies*. Trans. by Leslie W Barnard. Ancient Christian Writers. OCLC: 260002203. New York: Paulist Press, 1997. isbn: 978-0-8091-0472-7.
18. Sedar 'Olam (Polano, op. cit.)
19. Peter. *The Apocalypse of Peter*. Trans. by Alexander Roberts. url: `http://www.earlychristianwritings.com/text/apocalypsepeter-roberts.html` (visited on 08/29/2018).
20. M. Sabbat 4.11 (Polano, op. cit.)
21. B. Baba Batra 60b (ibid.)
22. B. Berakhot. 61b (ibid.)
23. Lampe, op. cit., p. 156.
24. Lawrence H Schiffman. *Who Was a Jew?: Rabbinic and Halakhic Perspectives on the Jewish-Christian Schism*. OCLC: 470701125. Hoboken, N.J.: Ktav Pub. House, 1985. isbn: 978-0-88125-053-4, pp. 155-6.

library committee members and officials

thomas adamo
anshu banerjee
ben goodrick-green
vanessa ip
adam mee
benedict randall shaw
arjun seth
jadd virji
jonathan watts
ben weiss
anna yang

chair jonny heywood
assistant chair isky mathews
editor joshua loo